# Real-time Constrained Nonlinear Optimization for Maximum Power Take-off of a Wave Energy Converter

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# Summary

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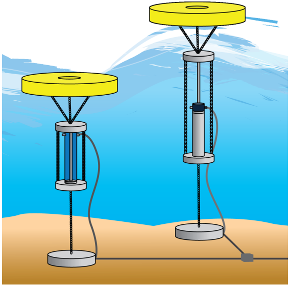
# Wave Energy Converters (WECs)



## Pros

- Green and emission-free
- Abundant and widely available
- Reliable and predictable

## Cons

- High installation and maintenance costs
- Impact on the marine ecosystem

# Dynamic model of a point-absorber wave energy converter

## Dynamic equations

$$m\,a(t) + r\,v(t) + k\,p(t) = F_R(t) + F_D(t) + u(t) + F_E(t)$$

### Remarks

- $m$ and $r$ and the mass and viscous dissipation of the device and mooring system
- $k$ is the hydrodynamic stiffness due to to the buoyancy force

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Radiation Force

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Nonlinear Drag Force

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Machinery Force

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## Dynamic equations

$$m\,a(t) + r\,v(t) + k\,p(t) = F_R(t) + F_D(t) + u(t) + F_E(t)$$

Excitation Force

### Remarks

- $m$ and $r$ and the mass and viscous dissipation of the device and mooring system
- $k$ is the hydrodynamic stiffness due to to the buoyancy force

# Hydrodynamic forces

$$F_R(t) = -m_\infty\, a(t) - \int_{-\infty}^{t} k_R(t-\tau)\, v(\tau)\, d\tau = -m_\infty\, a(t) - F_r(t)$$

$$F_D(t) = -\frac{1}{2} C_D\, \rho\, S\, (v(t))^2\, \mathrm{sgn}(v(t))$$

$$F_E(t) = \int_{-\infty}^{+\infty} k_E(t-\tau)\, \eta(\tau)\, d\tau$$

## Remarks

- $F_D(t)$ is a nonlinear and non-smooth function of the device velocity
- $m_\infty$ is the added mass, $k_R(t)$ is the causal radiation kernel
- $k_E(t)$ is the noncausal excitation kernel, $\eta(t)$ is the wave elevation at the device location

# State-space model

## State-space realization of $F_r(t)$

$$\dot{z}(t) = A_p z(t) + B_p v(t)$$
$$F_r(t) = C_p z(t) + D_P v(t)$$

## State-space WEC model

$$\dot{x} = A x(t) + f_D(x(t)) + B u(t) + E F_E(t)$$

### Remarks

- $u(t)$ is the control variable
- $F_E(t)$ is supposed to be known in advance or predicted through estimation techniques

# Summary

# MPC formulation

The goal is optimizing the WEC power take-off by maximizing the energy absorption over the interval $[t_0, t_0 + T]$:

$$\max E_a = \max - \int_{t_0}^{t_0+T} v(t)\, u(t)\, dt$$

subject to mechanical and dynamics constraints.
This reduces to the solution of the following nonlinear optimization problem:

$$\min \frac{1}{2} \int_{t_0}^{t_0+T} x^T S_v^T u + u^T S_v x \, dt$$

subject to

$$\begin{aligned} & x(t_0) = \bar{x}_0 \\ & \dot{x}(t) = Ax(t) + f_D(x(t)) + Bu(t) + EF_E(t), && t \in [t_0, t_0 + T] \\ & d(x(t), u(t)) \geq 0, && t \in [t_0, t_0 + T] \end{aligned}$$

# Problem discretization

The time interval is divided into $N$ shooting intervals. In each interval, the following conditions must be imposed:

$$
\begin{aligned}
& u_k(t) = u_k, && t \in [t_k, t_{k+1}] \\
& F_{Ek}(t) = F_{Ek}, && t \in [t_k, t_{k+1}] \\
& \dot{x}_k(t) = A x_k(t) + f_D(x_k(t)) + B u_k + E F_{Ek}(t), && t \in [t_k, t_{k+1}] \\
& x_k(t_{k+1}) = x_{k+1} && \\
& d(x_k, u_k) \geq 0 &&
\end{aligned}
$$

The nonlinear dynamic equations are discretized using an explicit fourth-order Runge-Kutta scheme

# Nonlinear Constrained Programming (1/2)

The discretized problem now appears as

$$\min \frac{1}{2} \sum_{k=0}^{N-1} x_k^T S_v^T u_k + u_k^T S_v x_k$$

subject to

$$x_0 = \bar{x}_0$$

$$x_k + \Delta t \sum_{i=1}^{s} b_i K_i^{RK} - x_{k+1} = 0, \qquad k \in [0, N-1]$$

$$K_i^{RK} = A\left(x_k + \Delta t \sum_{j=1}^{i-1} a_{ij} K_j^{RK}\right) + f_D\left(x_k + \Delta t \sum_{j=1}^{i-1} a_{ij} K_j^{RK}\right) + B u_k + E F_{E k}$$

$$d(x_k, u_k) \geq 0, \qquad k \in [0, N-1]$$

# Nonlinear Constrained Programming (2/2)

Define:

$$w = [u^T \quad x^T]^T$$

The optimization problem becomes:

$$\begin{aligned} &\min J(w) = \min \frac{1}{2} w^T H w \\ &\text{subject to} \\ &\quad C(w) + \Lambda \bar{x}_0 = 0 \\ &\quad D(w) \geq 0 \end{aligned}$$

where:

$$H = \begin{bmatrix} 0 & \bar{S}_v \\ \bar{S}_v^T & 0 \end{bmatrix}$$

The bar sign accounts for the discretization of the time integral

# Sequential Quadratic Programming (SQP)

- Define an initial guess for the optimization variable and the Lagrange multipliers $(w_0, \lambda_0, \mu_0)$
- At each iteration $k$, starting from $(w_k, \lambda_k, \mu_k)$, solve the following quadratic programming problem (QP):

$$\min_{\Delta w, \lambda, \mu} \frac{1}{2}\Delta w^T B_k \Delta w + b_k^T \Delta w$$

subject to

$$\nabla C|_{w_k} \Delta w + C|_{w_k} = 0$$
$$\nabla D|_{w_k} \Delta w + D|_{w_k} \geq 0$$

where $b_k = \nabla J(w)|_{w_k}$ is the gradient of the cost function and $B_k$ is the Hessian of the associated Lagrangian function:

$$B_k = H - \sum_i \lambda_i \nabla^2 C_i|_{w_k} - \sum_j \mu_j \nabla^2 D_j|_{w_k}$$

- Perform the update $(w_{k+1}, \lambda_{k+1}, \mu_{k+1}) = (w_k + \alpha \Delta w^*, \lambda^*, \mu^*)$
- Repeat until convergence

# Implementation features

- Gradients can be computed analytically or numerically. Many numerical approaches are available:
    - Automatic Differentiation
    - Adjoint Gradient
    - Finite Differences
    - Complex Derivative

  For the WEC implementation all gradients are computed analytically. The Hessian of the Lagrangian is calculated analytically as well
- A full step implementation, i.e. $\alpha = 1$, is considered
- The Hessian of the cost function is indefinite
- Inequality constraints involve motion constraints of the device and saturation constraints of the actuator:

$$
\begin{aligned}
u_{\min} &\leq u_k \leq u_{\max} \\
p_{\min} &\leq S_p x_k \leq p_{\max} \\
v_{\min} &\leq S_v x_k \leq v_{\max}
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# Handling the drag term

The calculation of the gradient of the equality constraints requires differentiability of the dynamic equations. The drag force, due to the sgn function is non-smooth:

$$F_D(t) = -\frac{1}{2} C_D \rho S (v(t))^2 \operatorname{sgn}(v(t))$$

A smooth approximation of the drag term is:

$$F_D(t) = -\frac{1}{2} C_D \rho S (v(t))^2 \tanh(K v(t))$$

where $K$ is a parameter governing the degree of smoothness. The dynamic equations are now twice differentiable

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# Condensed SQP

## Considerations

- The KKT matrix arising from the QP problem has a very sparse nature
- This sparsity can be exploited by projecting the cost function to the null space of the equality constraints
- This allows to remove the dependent variable $\Delta x$, which accounts for the state perturbation, and solve an optimization problem in $\Delta u$ only
- This reduces the optimization space and produces a dense KKT matrix

# Condensed SQP

Looking at the structure of $\nabla C|_{w_k}$:

$$\nabla C|_{w_k} = \begin{bmatrix} & & -I & & & \\ I+\Delta t B & & I+\Delta t \sum_i^s b_i \nabla_{x_0} K_i^{RK}|_{w_k} & -I & & \\ & \ddots & & \ddots & \ddots & \\ & I+\Delta t B & & & I+\Delta t \sum_i^s b_i \nabla_{x_{N-1}} K_i^{RK}|_{w_k} & -I \end{bmatrix}$$

Through an appropriate permutation matrix $P$, it is possible to rewrite the gradient as

$$P\nabla C|_{w_k} = \begin{bmatrix} L & -I \end{bmatrix}$$

This allows to rewrite the equality constraint in the QP iteration as:

$$\Delta x = L\Delta u + PC|_{w_k}$$

Substituting this relationship everywhere gives the reduced dense QP problem:

$$\min \frac{1}{2} u^T H_u u$$

subject to

$$d_u(u) \geq 0$$

$H_u$ is now positive definite, hence convergence is ensured

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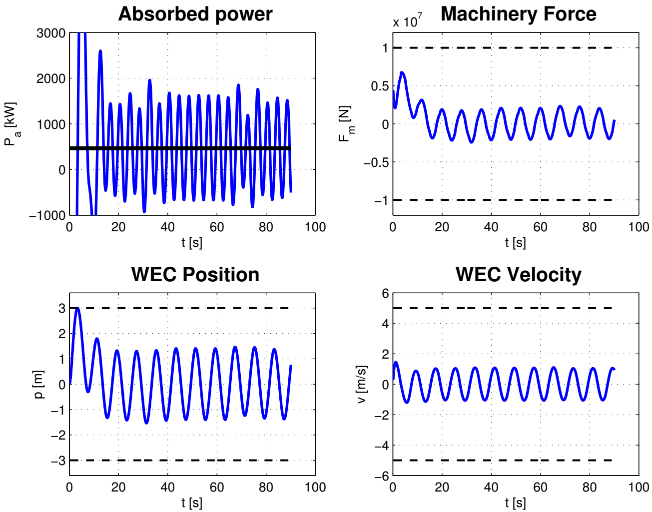
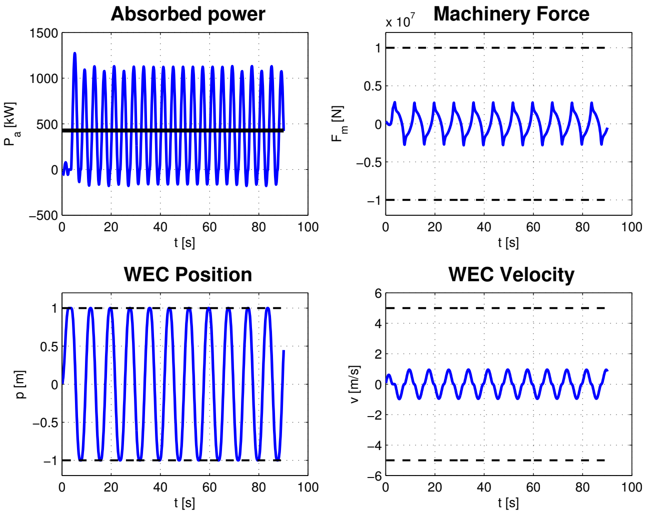
# Results for sine wave with $H_{m0} = 2m$, $T_0 = 8s$, $T = 10s$ (1/3)



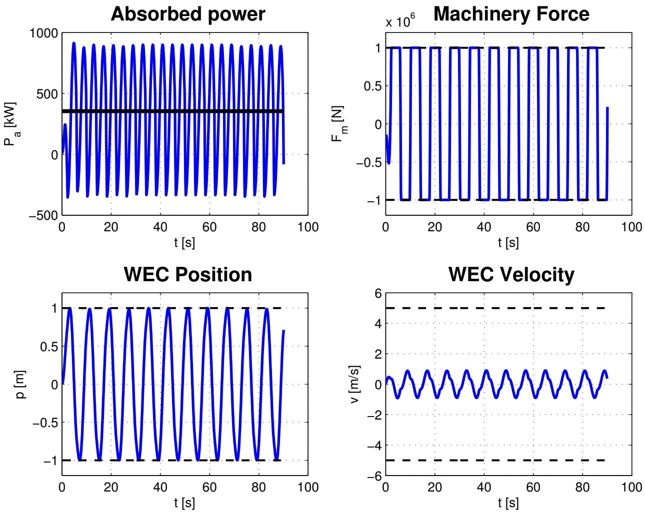

Constraints are $|u| \leq 10^7 N$, $|p| \leq 3m/s$, $|v| \leq 5m/s$

# Results for sine wave with $H_{m0} = 2m$, $T_0 = 8s$, $T = 10s$ (2/3)



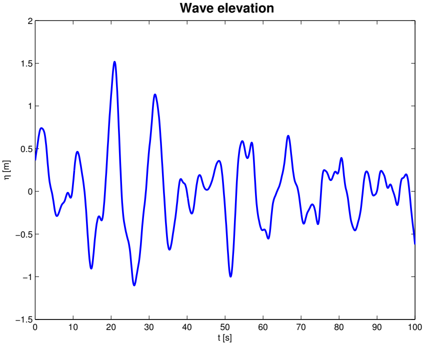

Constraints are $|u| \leq 10^7 N$, $|p| \leq 1 m/s$, $|v| \leq 5 m/s$

# Results for sine wave with $H_{m0} = 2m$, $T_0 = 8s$, $T = 10s$ (3/3)




Constraints are $|u| \leq 10^6 N$, $|p| \leq 1 m/s$, $|v| \leq 5 m/s$

# Results for JONSWAP sea spectrum with $H_{m0} = 2m$, $T_0 = 8s$, $T = 10s$

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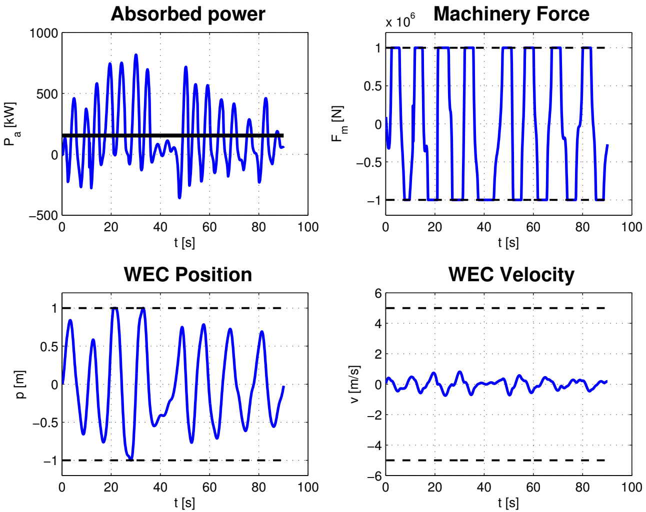


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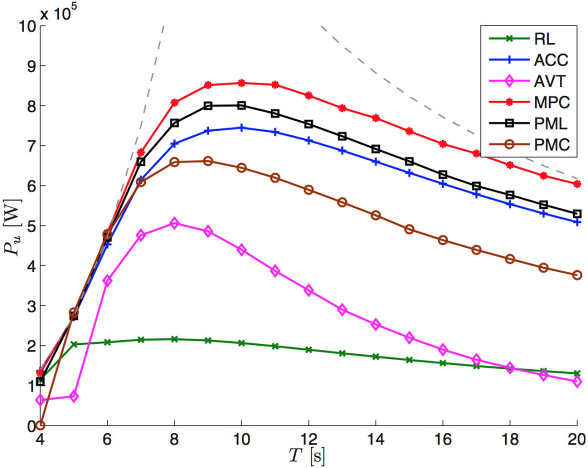
# Budal's diagram for sine waves with $H_{m0} = 2m$, $T = 10s$




*Legend:* RL = Resistive Loading, ACC = Approximate Complex-Conjugate Control, AVT = Approximate Optimal Velocity Tracking, MPC = Model Predictive Control, PML = Peak-Matching Latching Control, PMC = Peak-Matching Clutching Control

# Summary

# Conclusions

- Direct multiple shooting provides a fast way of solving constrained MPC problems involving nonlinear systems
- The condensing step further accelerates the solution of the QP problem
- Nonlinear MPC applied to the maximization of power take-off of a wave energy converter allows to approach the theoretical limit while outperforming other techniques
- This approach can easily be extended to any other WEC configuration and constraints