Exercise 6.1. Here we consider the model PDE

\[-\nabla \cdot (p\nabla u) + q u = f \quad \text{in } \Omega = (0, 1)^3\]
\[u = 0 \quad \text{on } \partial \Omega\]

where \(p_{\text{max}} \geq p(x, y, z) \geq p_{\text{min}} > 0\) in \(\Omega\), and \(q_{\text{max}} \geq q(x, y, z) \geq q_{\text{min}} \geq 0\) in \(\Omega\). Let

\[h = 1/(n + 1)\]
\[x_i = ih, \quad 0 \leq i \leq n + 1\]
\[y_j = jh, \quad 0 \leq j \leq n + 1\]
\[z_k = kh, \quad 0 \leq k \leq n + 1\]
\[U_{ijk} \approx u(x_i, y_j, z_k)\]

1. Find the 7-point finite difference approximation for this problem. Be sure to explain how the Dirichlet boundary conditions enter.

2. Compute the local truncation error for this approximation, which should be \(O(h^2)\). (Computing just the leading term exactly is sufficient.)

3. Find the \(N \equiv n \times n \times n\) linear system \(AU = F\). Use a natural ordering of the gridpoints. This will lead to a block \(n \times n\) tridiagonal matrix with \(n^2 \times n^2\) blocks. These blocks will themselves be block \(n \times n\), with \(n \times n\) blocks.

4. Let \(A\) be the \(N \times N\) matrix for the model problem and \(B\) the \(N \times N\) matrix for the model problem in the special case \(p(x, y, z) \equiv 1\) and \(q(x, y, z) \equiv 0\). Prove there exist constants \(C_1\) and \(C_2\), that don’t depend on \(h\), such that

\[0 < C_1 \leq \frac{x^t A x}{x^t B x} \leq C_2 < \infty.\]

(Follow the proof for two dimensions given in class.)

5. Using part 4, prove

\[\|U - u\|_{0,h} \leq C h^2 \|u\|_{4,h}\]