Exercise 5.1. In this problem, we will look at the 1-dimensional hierarchical basis for the space of continuous piecewise quadratic polynomials on a (non-uniform) mesh $T$. Let $\mathcal{M} = V \oplus W$, where $\mathcal{M}$ is the space of continuous piecewise quadratic polynomials, $V$ is the space of continuous piecewise linear polynomials, and $W$ is the space of piecewise quadratic “bump” functions. Take the reference element to be the unit interval $[0, 1]$.

a. Write down the quadratic nodal basis functions for the reference element. Write down the quadratic hierarchical basis functions for the reference element.

b. Let $a(u, v) = \int u'v' + uv \, dx$ be the inner product of interest. Compute the constant $\gamma$ in the strengthened Cauchy inequality

$$|a(v, w)| \leq \gamma \|v\| \|w\|$$

for all $v \in V$ and all $w \in W$. In particular, reduce the problem to a calculation on the reference element, and verify that $\gamma$ does not depend on the sizes on the elements in $T$.

Exercise 5.2. Let $T_c$ denote a coarse 1-dimensional mesh, and $T_f$ the mesh generated by the bisection of all elements in $T_c$. Let $\mathcal{M}_c$ denote the space of continuous piecewise linear polynomials on $T_c$ and $\mathcal{M}_f$ denote the space of continuous piecewise linear polynomials on $T_f$. We will consider the hierarchical decomposition $\mathcal{M}_f = V \oplus W$, where $V = \mathcal{M}_c$ and $W$ is the span of the nodal basis functions corresponding to the refined nodes in $T_f$. Take the reference element on the coarse space to be the unit interval $[0, 1]$.

a. Write down the nodal basis functions for $\mathcal{M}_f$ for the reference element. Write down the hierarchical basis functions for $\mathcal{M}_f$ for the reference element.

b. As above, let $a(u, v) = \int u'v' + uv \, dx$ be the inner product of interest. Compute the constant $\gamma$ in the strengthened Cauchy inequality

$$|a(v, w)| \leq \gamma \|v\| \|w\|$$

for all $v \in V$ and all $w \in W$. 