Exercise 3.1. In this exercise, we will consider the model two point boundary value problem:

\[-u'' = f\]

with \(u(0) = \alpha\) and \(u(1) = \beta\). We will discretize this problem using finite differences. The matrix involved will always arise from the standard three point centered difference approximation. Let \(h = 1/(n + 1)\), with \(x_i = ih, 0 \leq i \leq n + 1\). The difference equations have the form

\[-U_{i-1} + 2U_i - U_{i+1} = h^2 F_i\]

for \(1 \leq i \leq n\), with \(U_0 = \alpha, U_{n+1} = \beta\). The matrix system is of the form \(AU = F\) with \(A\) the \(n \times n\) tridiagonal matrix

\[
A = \frac{1}{h^2} \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& & \ddots & \ddots & \ddots \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{pmatrix}.
\]

To be able to compute the error, we can choose problems for which we know the exact solution. If the continuous solution \(u(x)\) is known, we can compute \(E_i = u(x_i) - U_i\). A simple choice is \(u(x) = x^p\), for which \(f(x) = -p(p-1)x^{p-2}\), \(\alpha = 0, \beta = 1\).

1. The classic second order discretization has

\[
F_1 = f(x_1) + \frac{\alpha}{h^2},
\]

\[
F_i = f(x_i) \quad 2 \leq i \leq n - 1
\]

\[
F_n = f(x_n) + \frac{\beta}{h^2}
\]

For this special problem, the cases \(p = 0, 1, 2, 3\) should give exact answers up to roundoff error, so you can use these cases to check the correctness of your program. Once you are satisfied you have a correctly working code, solve \(AU = F\) for some values of \(p\) that give smooth solutions, e.g. \(p = 4, 5, 6\). Use several values of \(n\), e.g. \(n_k = 2^k - 1\) for \(k = 4, 5, \ldots, 10\). Then \(h_k = 2^{-k}\). For \(O(h^2)\) second order convergence, you should see
the error decrease by a factor of approximately 4 for each reduction in $h$ by a factor of 2. You can observe this by computing the discrete norm

$$\|E_h\|^2 \equiv h \sum_{i=1}^{n} E_i^2$$

and observing $\|E_h\|$ decrease by an approximate factor of 4 as $n$ increases by a factor of approximately 2.

$$\frac{\|E_{2h}\|}{\|E_h\|} \approx 4.$$

(Be sure to take the square root; $\|E_h\|^2$ will decrease by a factor of approximately 16.) If $\|E_h\| \sim C h^p$, the order of convergence $p$ can be estimated from

$$\text{order} \approx \log_2 (\|E_{2h}\|/\|E_h\|).$$

2. When you get the correct behavior for your program in part 1, repeat the experiment above for some cases where the solution is NOT smooth. For example, try $p = 5/2, 3/2, 1/2$. What is the observed order of convergence?

3. Now try the order 6 compact scheme derived in class

$$F_1 = \frac{1}{24} \left\{ 5f(x_1 - \sqrt{2/5}h) + 14f(x_1) + 5f(x_1 + \sqrt{2/5}h) \right\} + \frac{\alpha}{h^2}$$

$$F_i = \frac{1}{24} \left\{ 5f(x_i - \sqrt{2/5}h) + 14f(x_i) + 5f(x_i + \sqrt{2/5}h) \right\} \quad 2 \leq i \leq n - 1$$

$$F_n = \frac{1}{24} \left\{ 5f(x_n - \sqrt{2/5}h) + 14f(x_n) + 5f(x_n + \sqrt{2/5}h) \right\} + \frac{\beta}{h^2}$$

For this choice of $F$ you should get exact answers up to roundoff for $p = 0, 1, 2, 3, 4, 5, 6, 7$, so you can use these values of $p$ to check the correctness of your code. Once you are satisfied you have a correctly working code, solve $AU = F$ for some values of $p$ that give smooth solutions, e.g. $p = 8, 9, 10$. Repeat the convergence study you used for parts 1-2. For $O(h^6)$ sixth order convergence, you should see the error decrease by a factor of approximately 64 for each reduction in $h$ by a factor of 2. Depending on the precision of the arithmetic that you use, you may see roundoff error begin to play a significant role for the larger values of $n$, so keep this point in mind as you analyze your results.

4. Now repeat the experiments of part 2 for the compact discretization scheme. What is the observed order of convergence?