Exercise 2.1. In this computer problem we will numerically verify the stability of the forward and backward Euler methods. We will solve the heat equation given by

\[ u_t = u_{xx} \]

for \( 0 < x < 1 \) and \( t > 0 \). The boundary conditions are \( u(0, t) = u(1, t) = 0 \) for all \( t > 0 \), and the initial condition is \( u(x, 0) = f(x) \), \( 0 \leq x \leq 1 \).

Let \( h = 1/8 \) \( (n = 7) \), and let the initial condition be \( f(x) = \sin \pi x \). The true solution will then be

\[ u(x, t) = e^{-\pi^2 t} \sin \pi x \]

a. For \( \Delta t = 1/4, 1/16, 1/64, 1/256, 1/1024 \), Compute the numerical solution for \( 0 \leq t \leq 5 \) using the forward Euler method. Print out the solution at \( t = 1, 2, 3, 4, 5 \).

b. What conclusions can you make about the stability of the forward Euler method?

c. Can you make any guess as to the effect of roundoff error on the computation?

d. For \( \Delta t = 1/4, 1/16, 1/64, 1/256, 1/1024 \), Compute the numerical solution for \( 0 \leq t \leq 5 \) using the backward Euler method. Note that you must solve a tridiagonal set of linear equations at each time step. Print out the solution at \( t = 1, 2, 3, 4, 5 \).

e. What conclusions can you make about the stability of the backward Euler method? Is the extra computational expense of the backward Euler method justified? Why?