Exercise 2.1. Consider the two point boundary value problem
\[-u'' + \gamma u = f\]
for $0 < x < 1$, with $u(0) = u(1) = 0$, and $\gamma \geq 0$ a scalar constant. Find a compact fourth order discretization of the form
\[A u(x) + B u(x + h) + C u(x - h) = D f(x) + E f(x + h) + F f(x - h)\]
where $A, B, \ldots, F$ are constants (depending on $h$ and $\gamma$).

Exercise 2.2. Consider the 2 point boundary value problem with periodic boundary conditions:
\[-u'' + u = f\]
for $0 < x < 1$, with $u(0) = u(1)$, $u'(0) = u'(1)$. We will discretize this problem on a uniform mesh of size $h = 1/n$, $x_i = ih$, $0 \leq i \leq n$.

1. Using a second order centered finite difference approximation, derive the set of linear equations $A_2 U_2 = F_2$. The matrix $A_2$ will be an “almost tridiagonal” circulant matrix.

2. Using a fourth order centered finite difference approximation, derive the set of linear equations $A_4 U_4 = F_4$. The matrix $A_4$ will be an “almost pentadiagonal” circulant matrix.

3. Analyze the defect correction scheme:
\[
\begin{align*}
A_2 U_2 &= F_2 \\
A_2 E &= F_4 - A_4 U_2 \\
\hat{U} &= U_2 + E
\end{align*}
\]
In particular show that $\hat{U}_i = u(x_i) + O(h^4)$. 