**Exercise 1.1.** Consider the Stokes equations in two dimensions. The bilinear form is

\[ B(u, v) = a(u, v) - b(v, p) - b(u, q) = (f, v) \]

where

\[ a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega \nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 \, dx \]

and

\[ b(u, q) = \int_\Omega \nabla \cdot u q \, dx \]

for \( u, v \in H^1_0 \times H^1_0 \) and \( p, q \in L^2_0 \). We discretize on a shape regular, quasi uniform, triangular mesh, using the mini-element: continuous piecewise linear finite elements with cubic bubble functions \( \tilde{P}^1_1 \times \tilde{P}^1_1 \subset H^1_0 \times H^1_0 \), and continuous piecewise linear finite elements with average value zero \( \tilde{P}^1_1 \subset L^2_0 \).

1. On triangle \( t \), let \( u \) be a linear polynomial and \( v \) be the cubic bubble function with support on \( t \). Show:

\[ \int_t \nabla u \nabla v \, dx = 0 \]

The identity

\[ \int_t c^m_1 c^2_2 c^k_3 = \frac{m! n! k! 2|t|}{(m + n + k + 2)!} \]

is helpful. Here the \( c_i \) are baricentric coordinates (linear basis functions) on \( t \) and \( v = 27 c_1 c_2 c_3 \). Remember \( c_1 + c_2 + c_3 = 1 \) and \( \nabla c_1 + \nabla c_2 + \nabla c_3 = 0 \).

2. Show that the block \( 2 \times 2 \) KKT system

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
U \\
P
\end{pmatrix}
= 
\begin{pmatrix}
F^U \\
0
\end{pmatrix}
\]

can be written as a block \( 5 \times 5 \) system

\[
\begin{pmatrix}
A_1 & 0 & 0 & 0 & B_1^T \\
0 & D_1 & 0 & 0 & B_1^T \\
0 & 0 & A_2 & 0 & B_2^T \\
0 & 0 & 0 & D_2 & B_2^T \\
B_1 & B_1 & B_2 & B_2 & 0
\end{pmatrix}
\begin{pmatrix}
U_1 \\
\tilde{U}_1 \\
U_2 \\
\tilde{U}_2 \\
P
\end{pmatrix}
= 
\begin{pmatrix}
F_1 \\
\tilde{F}_1 \\
F_2 \\
\tilde{F}_2 \\
0
\end{pmatrix}
\]

where the \( A_i \) are associated with the piecewise linear parts of the velocity, and the \( D_i \) are diagonal matrices associated with the cubic bubble functions.
3. Use static condensation (i.e., block Gaussian elimination) to reduce this system to the block $3 \times 3$ system

$$
\begin{pmatrix}
A_1 & 0 & B_1^T \\
0 & A_2 & B_2^T \\
B_1 & B_2 & -C
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
P
\end{pmatrix}
=
\begin{pmatrix}
F_1 \\
F_2 \\
P
\end{pmatrix}
$$

where $C$ is symmetric and positive semi-definite. This reduced Schur complement system is the one usually solved in practice. The bubble function part of the solution is not computed; the bubble functions were introduced mainly for stability and not for accuracy.

4. Show that the reduced system is actually a special case of the Petrov-Galerkin discretization studied in class (i.e., the matrix $C$ and the right hand side $G$ are equal to the corresponding terms in the Petrov-Galerkin formulation for a particular choice of constants). Thus the mini-element and the Petrov-Galerkin formulation stabilize the Stokes equations in a similar fashion.