# Math 270B: Numerical Approximation and Nonlinear Equations

Instructor: Michael Holst

Winter Quarter 2017

PRACTICE/SAMPLE FINAL EXAM

| | | |
|---|---|---|
| #1 | 10 | |
| #2 | 10 | |
| #3 | 10 | |
| #4 | 10 | |
| #5 | 10 | |
| #6 | 10 | |
| #7 | 10 | |
| #8 | 10 | |
| Total | 80 | |

Name ________________________________________

Instructions:

- No calculators allowed (or needed).
- No crib sheets or notes allowed.
- You must show your work.
- Put your name on the line above and staple this sheet to the pages containing your exam solutions.

Questions:

1. (Calculus in $\mathbb{R}^n$: 10 points.) Calculate the derivative $f'(x)$, the gradient $g(x) = \nabla f(x)$, and the Hessian matrix $H(x)$ (the Jacobian matrix of $\nabla f(x)$) of the following real-valued function of $n$ real variables:
$$f(x) = \frac{1}{2}x^T Ax - x^T b, \quad \text{where: } A \in \mathbb{R}^{n\times n},\ x, b \in \mathbb{R}^n.$$
Show your work; as usual, I recommend using the Gateaux variation.

2. (Taylor Expansion in $\mathbb{R}^n$ and Newton's Method: 10 points.)
   (a) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable. Using the fundamental theorem of calculus in $\mathbb{R}^n$, derive the following Taylor expansion with integral remainder:
   $$F(x + h) = F(x) + F'(x)h + \int_0^1 \{F'(x + \xi h) - F'(x)\}\, h\ d\xi.$$
   (b) Using this Taylor expansion, derive Newton's method for $F(x) = 0$, where $F : \mathbb{R}^n \to \mathbb{R}^n$.
   (c) Give a complete algorithm (in pseudocode only) for implementing Newton's method that you just derived in (b) on a computer. Include backtracking line-search (i.e., damping) and allow for inexact solves of the linearized systems at each step.

3. (Newton's Method and Unconstrained Optimization: 10 points.)
   (a) Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be analytic (have a convergent Taylor expansion at all points in $\mathbb{R}^n$). Prove that a descent direction $p \in \mathbb{R}^n$ for $f(x)$ at a point $x \in \mathbb{R}^n$ is always a direction of decrease for $f(x)$ at $x$.
   (b) Let $F(x) : D \subset \mathbb{R}^n \to \mathbb{R}^n$, and assume that the Jacobian $F'(x) : D \subset \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is Lipschitz continuous with Lipschitz constant $\gamma$. Show that the error in the linear model
   $$L_k(x) = F(x^k) + F'(x^k)(x - x^k)$$
   of $F(x)$ can be bounded as follows:
   $$\|F(x) - L_k(x)\| \le \frac{1}{2}\gamma\|x - x^k\|^2.$$

4. (Constrained Optimization: 10 points)
Let $f, C\colon \mathbb{R}^2 \to \mathbb{R}$, with $f(x) = 3x_2 + x_1^2 + x_2^2$, and $c(x) = x_1^2 + (x_2 + 1)^2 - 1 = 0$.
   (a) Form the Lagrangian $L(x, \lambda)$ from $f(x)$ and $c(x)$ that you will use to minimize $f(x)$ over $\mathbb{R}^2$, subject to $c(x) = 0$.
   (b) Find a point satisfying the (first order) KKT conditions. Verify (check second order conditions) that it is an optimal point.
   (c) Repeat Parts (a) and (b) with $f(x) = x_1^3 + x_2^3$.

5. (Polynomial forms and interpolation: 10 points.) Assume that we are given $n+1$ distinct points and corresponding function values:

$$\begin{array}{cccc} x_0 & x_1 & \cdots & x_n \\ f(x_0) & f(x_1) & \cdots & f(x_n) \end{array}$$

   (a) Write down the unique degree $n$ interpolating polynomial $p_n(x)$ that interpolates $f(x)$ at the $n+1$ points.

   (b) Now write down the Newton form of this polynomial.

   (c) Write down the expression for interpolation error.

6. (Polynomial Interpolation Construction: 10 points.)

   (a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates the data:

| x | f(x) |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 13 |

   (b) If the function $f(x)$ that generated the above data was actually the cubic polynomial $P_3(x) = 2x^3 - x^2 + 1$, derive an error bound (a fairly "tight" one) for the interval $[0, 2]$.

7. (Difference Approximations to Derivatives: 10 points.)

   (a) Show that the forward difference formula has an error expansion of the form:

$$f'(x) = \left[\frac{f(x+h) - f(x)}{h}\right] - \frac{f''(x)}{2}h - \frac{f'''(x)}{6}h^2 + O(h^3).$$

   (b) Derive an expression for the error in the following difference approximation to $f'(x)$:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

   (c) Derive an expression for the error in the following difference approximation to $f''(x)$:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

8. (Quadrature: 10 points.)

   (a) Use the (non-composite) Trapezoid Rule to approximate the following integral:

$$\int_0^1 x^3 dx,$$

   and derive an error bound. Also, compare your numerical result with the exact integral.

   (b) Determine the number of internals $n$ required for use with the *composite* version of the Trapezoid rule in order to approximate the integral to five digits of accuracy (e.g., ensure that $|error| < 10^{-5}$).