MATH 270B: Numerical Approximation and Nonlinear Equations

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Homework Assignment #2
Due : Give to the class TA within two weeks if you would like it marked.

Our goal in this homework is to further study the approach of using a quadratic model as a basis for developing methods for unconstrained optimization, and to study the method of Lagrange multipliers for incorporating equality constraints into optimization.

The starred exercises are those that require the use of a tool such as MATLAB.

Exercise 2.1. Let H be a symmetric matrix with spectral decomposition $H = VDV^T$.

(a) Show that an eigenvector v associated with a positive eigenvalue λ satisfies $v^T H v > 0$.

(b) Write down the inverse of H in terms of V and D.

(c) If r is a positive integer, give an expression for H^r in terms of D and V. If H is positive definite, find a matrix B such that $H = B^2 = BB$ (B is the "square root" of H).

(d) Let α denote a scalar such that the matrix $H - \alpha I$ is nonsingular. If $\psi(\alpha)$ is the univariate function $\psi(\alpha) = u^T(H - \alpha I)^{-1}u$, where u is a nonzero vector, find $\psi'(\alpha)$.

Exercise 2.2. Given each of the following cases of a gradient $g(\bar{x})$ and Hessian $H(\bar{x})$ defined at a point $\bar{x}$, discuss the optimality of $\bar{x}$. I.e., check the first and second order conditions for optimality. (Do not use MATLAB, atleast for the 2x2 cases; you may need to know how to compute eigenvalues by hand for 2x2 cases in the final exam.)

(i) $$g(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

(ii) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}.$$

(iii) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$$

(iv) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

(v) $$g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Exercise 2.3.* Write a MATLAB function with specification `[f,g,H] = ex33(x)` that computes $f(x)$, $g(x)$ and $H(x)$ for the function

$$f(x) = e^{x_3} x_1^2 + 2x_2^2 + x_3^2 \cos x_1$$

at any point x. Use your function to compute $f(x)$, $g(x)$, and $H(x)$ at $x = (0,\ 0,\ 0)^T$ and $x = (-1,\ 2,\ -2)^T$. In each case, compute the spectral decomposition of the Hessian matrix and indicate if the necessary and sufficient conditions for unconstrained local minimization are satisfied.

Exercise 2.4. Let $q(x)$, $x \in \mathbb{R}^n$, be the quadratic function $q(x) = c^T x + \frac{1}{2} x^T H x$, where H is symmetric.

(a) Write down an expression for $\nabla q(x)$ in terms of c, H and x.

(b) Given an arbitrary point x_0 and a direction p, write down the Taylor-series expansion of $q(x_0 + p)$.

(c) For this part, consider $q(x)$ such that H is positive definite. If p is a direction such that $\nabla q(x_0)^T p < 0$, show that there exists a *positive* minimizer α^* of $q(x_0 + \alpha p)$. Derive a closed-form expression for α^*.

Exercise 2.5.* Write a MATLAB m-file `steepest.m` that implements the method of steepest descent with a backtracking line search. Your function must include the following features.

- Use $\mu = \frac{1}{4}$ to define the sufficient-decrease criterion in the backtracking algorithm.
- The minimization must be terminated when either $\|g(x_k)\| \le 10^{-5}$ or 75 iterations are performed. Any MATLAB "`while`" loop must include a test that will terminate the loop if it is executed more than 20 times.

Use `steepest.m` to find a minimizer of the function

$$f(x) = e^{x_3}x_1^2 + 2x_2^2 + x_3^2\cos x_1,$$

starting at the point $(-1,\ 1,\ 1)^T$ (first write a MATLAB function as in Exercise 2.3). Next, minimize the function (again, first write a MATLAB function as in Exercise 2.3)

$$f(x) = x_1 + x_2 + x_3 + x_4 + x_1^2 + x_2^2 + 10^{-1}x_3^2 + 10^{-3}x_4^2,$$

starting at the point $(-1,\ 0,\ 1,\ 1)^T$. Compare the two runs. Can you explain why steepest descent behaves like this?

Exercise 2.6.* Modify the MATLAB m-file `steepest.m` from Exercise 2.5 to produce a MATLAB m-file `newton.m` that implements the Newton's method for optimization with a backtracking line search. Repeat the two examples in Exercise 2.5 using the `newton.m` m-file.

Exercise 2.7. Consider the nonlinearly constrained problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{minimize}} & 3x_2 + x_1^2 + x_2^2 \\ \text{subject to} & x_1^2 + (x_2+1)^2 - 1 = 0. \end{array} \tag{2.1}$$

(a) Show that $x(\alpha) = (\sin\alpha,\ \cos\alpha - 1)^T$ is a feasible path for the nonlinear constraint $x_1^2 + (x_2+1)^2 - 1 = 0$ of problem (2.1). Compute the tangent to the feasible path at $\bar{x} = (0,\ 0)^T$.

(b) If $f(x)$ denotes the objective function of problem (2.1), find an expression for $f\,(x(\alpha))$ and compute $f\,(x(0))$.

(c) Define the *Lagrangian function* $L(x,\lambda)$ and *constraint Jacobian* $J(x)$ for problem (2.1). Derive $\nabla L(x,\lambda)$, the gradient of the Lagrangian, and $\nabla^2_{xx}L(x,\lambda)$, the Hessian of the Lagrangian with respect to x.

(d) Determine whether or not the point $\bar{x} = (0,\ 0)^T$ is a constrained minimizer of problem (2.1).

Exercise 2.8. Consider the problem

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{minimize}} & x_1^2 + 2x_2^2 \\ \text{subject to} & x_1 + x_2 - 1 = 0. \end{array}$$

(a) Find a point satisfying the KKT conditions. Verify that it is indeed an optimal point.

(b) Repeat Part (a) with the objective replaced by $x_1^3 + x_2^3$.

Exercise 2.9.* Write a MATLAB function that will compute $c(x)$ and $J(x)$ for the constraint function

$$c(x) = x_1 + x_2 - x_1x_2 - \frac{3}{2}.$$

Use your function to find $c(x)$ and $J(x)$ at $x = (.1,\ -.5)^T$, $x = (.5,\ -1)^T$ and $x = (1.18249728,\ -1.73976692)^T$.

At each of these points, discuss the optimality of the constrained minimization problem:

$$\begin{array}{ll} \underset{x\in\mathbb{R}^2}{\text{minimize}} & e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1) \\ \text{subject to} & x_1 + x_2 - x_1x_2 - \frac{3}{2} = 0. \end{array}$$

Exercise 2.10. Assume that $c : \mathbb{R}^n \to \mathbb{R}^m$, with $c_i \in C^1(\mathbb{R},\mathbb{R})$, $i = 1,\ldots,m$, and with $1 \le m \le n$. We know that if x is a feasible point, then $T^0(x) \subset N(J(x))$, where $T^0(x)$ is the tangent cone at x, $J(x) = \nabla c(x)$ is the Jacobian matrix of the constraints at x, and $N(J(x))$ is the nullspace of $J(x)$ at x. Prove that if the rows of $J(x)$ at x are linearly independent, then constraint qualification holds at x (i.e., $T^0(x) \equiv N(J(x))$).

Exercise 2.11. Prove the KKT Theorem (existence of Lagrange Multipliers when constraint qualification holds).