HW2.

3.15.2. $B=\begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ & 1 & & & \vdots \\ & & \ddots & & -1 \\ & & & & 1 \end{pmatrix}_{n\times n}$ $\det(B) = 1\times 1\times\cdots\times 1 = 1$

$K_\infty(B) = \|B\|_\infty \cdot \|B^{-1}\|_\infty$ $\|B\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |b_{ij}| = n$

$$\begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ & 1 & -1 & & \vdots \\ & & \ddots & & -1 \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & & \\ & 1 & 1 & \cdots & 1 \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} = I$$

$$\therefore B^{-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & & & \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & & \\ & 1 & 1 & \cdots & 1 \\ & & \ddots & & \\ & & & & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

focus on the first row, by the production of the above elementary matrices

$$(1,1,\cdots 1) \to (1,1,2,2\cdots 2) \to (1,1,2,4\cdots 4) \cdots \to (1,1,2,4\cdots 2^{n-2})$$

$\therefore \|B^{-1}\|_\infty = 1+1+2+\cdots 2^{n-2} = 2^{n-1}$ $\therefore K_\infty(B) = n\cdot 2^{n-1}$

3.15.3.

$$k(AB) = \|AB\|\cdot\|(AB)^{-1}\| \le \|A\|\cdot\|B\|\cdot\|B^{-1}\|\cdot\|A^{-1}\| = k(A)\cdot k(B)$$

3.15.4.

$A=\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ $A^{-1}=\begin{pmatrix} 1 & -\gamma \\ & 1 \end{pmatrix}$

$$\kappa_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = (1+r)\cdot(1+r) = (1+r)^2$$

$$\kappa_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1 = (1+r)\cdot(1+r) = (1+r)^2$$

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \le \kappa_\infty(A)\cdot \frac{\|\delta b\|_\infty}{\|b\|_\infty} = (1+r)^2 \cdot \frac{\|\delta b\|_\infty}{\|b\|_\infty}$$

the well-conditioning depends on the value of r

3.15.5. $$A = \begin{pmatrix} 1 & & & 1 \\ -1 & 1 & & \vdots \\ -1 & -1 & \ddots & 1 \\ -1 & -1 & \cdots & 1 \end{pmatrix}$$ Consider row operations

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & \\ & 1 & & \\ & \vdots & \ddots & \\ & 1 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{pmatrix} A = \begin{pmatrix} 1 & & & 1 \\ & 1 & & 2 \\ & & \ddots & 4 \\ & & & \vdots \\ & & & 2^{n-1} \end{pmatrix}$$

$\therefore$ A admits an LU factorization. $|l_{ij}| \le 1$ and $u_{nn} = 2^{n-1}$

3.15.9. $$b = \sum_{i=1}^{n} c_i v_i$$

$$Ax = A\left(\sum_{i=1}^{n} \frac{c_i}{\lambda_i} v_i\right) = \sum_{i=1}^{n} \frac{c_i}{\lambda_i} A v_i = \sum_{i=1}^{n} c_i v_i = b$$

3.15.10. $\begin{pmatrix} 1001 & 1000 \\ 1000 & 1001 \end{pmatrix}$ has eigenvalue and eigenvector

$$\lambda_1 = 1 \qquad V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 2001 \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

when $b = \begin{pmatrix} 2001 \\ 2001 \end{pmatrix} = \lambda_2 \cdot V_2 \qquad X = V_2$

$$\delta b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{V_1 + V_2}{2} \qquad \delta x = \frac{1}{2}\left(\frac{V_1}{\lambda_1} + \frac{V_2}{\lambda_2}\right)$$

$$\therefore \frac{\|\delta x\|}{\|X\|} \Big/ \frac{\|\delta b\|}{\|b\|} \approx \sqrt{\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}} \Big/ \frac{1}{\lambda_2} = \sqrt{\left(\frac{\lambda_2}{\lambda_1}\right)^2 + 1} \approx 2001$$

Since A is ill-conditioned.

when $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \delta x = \begin{pmatrix} 0.001 \\ 0 \end{pmatrix}$ regard $A^{-1} b = X$

which A^{-1} has eigenvalues $\frac{1}{\lambda_2}, \frac{1}{\lambda_1}$. and prove in the same way

3.15.14. $A = QR$

$$k_2(A) = \|QR\|_2 \cdot \|R^{-1}Q^T\|_2 = \|R\|_2 \cdot \|R^{-1}\|_2 = k_2(R)$$

$$k_1(A) = \|QR\|_1 \cdot \|R^{-1}Q^T\|_1$$

$$\|QR\|_1 = \max_{x \neq 0} \frac{\|QRx\|_1}{\|x\|_1} \cancel{\leq \max_{x \neq 0} \frac{\sqrt{n}\,\|QRx\|_2}{\|x\|_2}}$$

$$\cancel{= \max_{x \neq 0} \frac{\sqrt{n}\,\|Rx\|_2}{\|x\|_2} = \sqrt{n} \cdot \|R\|_2}$$

suppose $Q = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & - & a_{nn} \end{pmatrix}$ $\qquad Qx = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n \end{pmatrix}$

$$\|Qx\|_1 = |a_{11}x_1 + \cdots a_{1n}x_n| + \cdots |a_{n1}x_1 + \cdots a_{nn}x_n| \leq (|a_{11}| + |a_{21}| + \cdots |a_{n1}|) \cdot |x_1| + \cdots (|a_{1n}| + \cdots |a_{nn}|) |x_n|$$

$$\leq \sqrt{n} \cdot (|x_1| + |x_2| + \cdots |x_n|) = \sqrt{n} \cdot \|x\|_1$$

$$\therefore \quad \|QR\|_1 = \max_{x \neq 0} \frac{\|QRx\|_1}{\|x\|_1} \leq \max_{x \neq 0} \frac{\sqrt{n} \cdot \|Rx\|_1}{\|x\|_1} = \sqrt{n} \cdot \|R\|_1$$

$$\|R^{-1}Q^T\|_1 = \max_{x \neq 0} \frac{\|R^{-1}Q^Tx\|_1}{\|x\|_1} \leq \max_{x \neq 0} \frac{\sqrt{n} \cdot \|R^{-1}Q^Tx\|_1}{\|Q^Tx\|_1} = \max_{x \neq 0} \frac{\sqrt{n}\, \|R^{-1}x\|_1}{\|x\|_1} = \sqrt{n} \cdot \|R^{-1}\|_1$$

$$\therefore \quad k_1(A) \leq n \cdot \|R\|_1 \cdot \|R^{-1}\|_1 = n \cdot k_1(R)$$

Similarly $\quad R = Q^TA \qquad k_1(R) = k_1(Q^TA) \leq n \cdot k_1(A)$

4.8.2. Gauss-Seidel method corresponds to split $A = M - N$

$M = \begin{pmatrix} a_{11} & & & \\ a_{12} & a_{22} & & \\ & & \ddots & \\ a_{n1} & a_{n2} & - & a_{nn} \end{pmatrix} \qquad N = \begin{pmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & -a_{23} \cdots & -a_{2n} \\ & & \ddots & \\ 0 & \cdots & 0 & 0 \end{pmatrix}$

$M^{-1} \cancel{A} N x = \lambda x \qquad \lambda M x = N x \qquad \lambda \sum_{j=1}^{i} a_{ij} x_j = \sum_{j=i+1}^{n} (-a_{ij}) x_j$

suppose $|x_i| = \|x\|_\infty$ for some i.

$$\lambda(a_{i1}x_1+a_{i2}x_2+\cdots a_{ii}x_i)+(a_{ii+1}x_{i+1}+\cdots a_{in}x_n)=0$$

$$\lambda a_{ii}=\lambda\left(a_{i1}\cdot\frac{x_1}{x_i}+a_{i2}\cdot\frac{x_2}{x_i}+\cdots a_{ii-1}\cdot\frac{x_{i-1}}{x_i}\right)+\left(a_{ii+1}\cdot\frac{x_{i+1}}{x_i}+\cdots a_{in}\cdot\frac{x_n}{x_i}\right)$$

$$|\lambda|\cdot|a_{ii}|\le|\lambda|\cdot(|a_{i1}|+|a_{i2}|+\cdots|a_{ii-1}|)+|a_{ii+1}|+\cdots|a_{in}|$$

$$\therefore\ |\lambda|\le\frac{|a_{ii+1}|+\cdots|a_{in}|}{|a_{ii}|-(|a_{i1}|+\cdots|a_{ii-1}|)}<1$$

Since A is strictly diagonally dominant

4.8.3

$$\alpha\cdot\sin(kj\theta)-(\sin((k-1)j\theta)+\sin((k+1)j\theta))$$

$$=\alpha\cdot\sin(kj\theta)-2\sin(kj\theta)\cos(j\theta)=(\alpha-2\cos(j\theta))\cdot\sin(kj\theta)$$

$\therefore\ (\sin(j\theta),\ \sin(2j\theta)\ \cdots\ \sin(nj\theta))^T$ has eigenvalue $\alpha-2\cos(j\theta)$