Answers for HW 1

8. transfer A to an upper triangular matrix by similar transformation

$$P^{-1}AP = R = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

$$A = P\begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}P^{-1}$$ simple calculation shows $$A^k = P\begin{pmatrix} \lambda_1^k & & * \\ & \lambda_2^k & \\ & & \ddots \\ & & & \lambda_n^k \end{pmatrix}P^{-1}$$

$$f(A) = P\begin{pmatrix} f(\lambda_1) & & * \\ & f(\lambda_2) & \\ & & \ddots \\ & & & f(\lambda_n) \end{pmatrix}P^{-1}$$ f is polynomial function

$\therefore$ $f(A)$ has eigenvalues $\{f(\lambda_1), f(\lambda_2) \cdots f(\lambda_n)\}$ $\qquad \lambda(f(A)) = f(\lambda(A))$

in particular A^2 has eigenvalues $\{\lambda_1^2, \lambda_2^2 \cdots \lambda_n^2\}$

$$\therefore \rho(A^2) = \max|\lambda_i^2| = (\max|\lambda_i|)^2 = \rho(A)^2$$

10. Insufficient condition to prove that if A, B share the same eigenvectors, then $AB = BA$.

(add condition that A, B are diagonalizable)

which means the eigenvectors could span the whole space

then for any eigenvector v,

$$ABv = A(\lambda_B v) = \lambda_B (Av) = \lambda_B \lambda_A v = BAv$$

$\therefore (AB - BA)v = 0$ for $\forall v \in R^n$ $\quad \therefore AB = BA$

13. lemma: $\|UAV\|_F = \|A\|_F$ where U, V are orthogonal matrix

suppose A has SVD decomposition $A = U\Sigma V^T$

$$\|AX - I_m\|_F = \|U\Sigma V^T X - I_m\|_F = \|U(\Sigma V^T X - U^T)\|_F$$

$$= \|\Sigma(V^T X) - U^T\|_F$$

$$\left\| \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ & & & \end{pmatrix} \begin{pmatrix} & V^T X & \end{pmatrix} - \begin{pmatrix} & U^T & \end{pmatrix} \right\|_F$$

$\|\Sigma V^T X - U^T\|_F$ is minimized iff the first n rows of $\Sigma V^T X$ coincide with first n rows of U^T, this generates a unique choice for $V^T X = \begin{pmatrix} 1/\sigma_1 & & & \\ & 1/\sigma_2 & & \\ & & \ddots & \\ & & & 1/\sigma_n \end{pmatrix} \begin{pmatrix} & U^T & \end{pmatrix}$

$$X = V \begin{pmatrix} 1/\sigma_1 & & & \\ & 1/\sigma_2 & & \\ & & \ddots & \\ & & & 1/\sigma_n \end{pmatrix} U^T = A^\dagger$$

17. $$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$$

$$\|A\|_1 = \sup \frac{\|Ax\|_1}{\|x\|_1} \le \sup \frac{\sqrt{n}\cdot\|Ax\|_2}{\|x\|_2} = \sqrt{n}\cdot\|A\|_2$$

$$\|A\|_2 = \sup \frac{\|Ax\|_2}{\|x\|_2} \le \sup \frac{\|Ax\|_1}{\frac{1}{\sqrt{n}}\|x\|_1} = \sqrt{n}\cdot\|A\|_1$$

$$\|x\|_\infty \le \|x\|_1 \le n\,\|x\|_\infty \qquad \|x\|_\infty \le \|x\|_2 \le \sqrt{n}\cdot\|x\|_\infty$$

prove redation of $\|A\|_1$ and $\|A\|_\infty$. $\|A\|_2$ and $\|A\|_\infty$ in the same way. $A=(a_1, a_2 \cdots a_n)$

$$\|A\|_1 = \max_{j=1}^{n} \|a_j\|_1 \le \max_{j=1}^{n} \sqrt{n}\,\|a_j\|_2 \le \sqrt{n}\,\|A\|_F$$

$$\|A\|_2 = \sigma_1 \le \sqrt{\sigma_1^2+\sigma_2^2+\cdots\sigma_n^2} = \|A\|_F$$

$$\|A\|_\infty = \|A^T\|_1 \le \sqrt{n}\,\|A^T\|_F = \sqrt{n}\cdot\|A\|_F$$

$$\|A\|_F = \sqrt{\sigma_1^2+\sigma_2^2+\cdots\sigma_n^2} \le \sqrt{n}\cdot\sigma_1 = \sqrt{n}\cdot\|A\|_2$$

$$\|A\|_F = \sqrt{\|a_1\|_2^2+\|a_2\|_2^2+\cdots\|a_n\|_2^2} \le \sqrt{\|a_1\|_1^2+\|a_2\|_1^2+\cdots\|a_n\|_1^2}$$

$$\le \sqrt{n}\cdot\|A\|_1$$

$$\|A\|_F = \|A^T\|_F \le \sqrt{n}\,\|A^T\|_1 = \sqrt{n}\,\|A\|_\infty$$

2.4. $X(x_0, t) = x_0 e^{at} \omega(t)$

$$\kappa_{abs}(x_0) = \left| \frac{\partial X}{\partial x_0} \right| = \left| e^{at} \omega(t) \right| \le e^{at}$$

on unbounded interval $\quad a>0 \quad \lim_{t\to\infty} e^{at} = \infty \quad$ ill conditioned

$\quad a<0 \quad \lim_{t\to\infty} e^{at} = 0 \quad$ well conditioned