Math 270A: Numerical Linear Algebra

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PRACTICE/SAMPLE FINAL EXAM

#1	10	
#2	10	
#3	10	
#4	10	
#5	10	
#6	10	
Total	60	

Name ____

Instructions:

- No calculators allowed (or needed).
- No crib sheets or notes allowed.
- You must show your work.
- Put your name on the line above and staple this sheet to the pages containing your exam solutions.

Questions:

1. (*Basic Linear Algebra*)

   Let V be a normed space. Recall that two norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ on V are called *equivalent* if:

   $$C_1\|u\|_X \leq \|u\|_Y \leq C_2\|u\|_X, \quad \forall u \in V.$$

   In the case of a finite-dimensionals space V, we noted that all norms can be shown to be equivalent. Consider now the specific finite-dimensional space $V = \mathbb{R}^n$.

   (a) Determine constants C_1 and C_2 for the l^p norms for $p = 1, 2, \infty$. In particular, show the following tight bounds:
   $$\|u\|_\infty \leq \|u\|_2 \leq \|u\|_1 \leq \sqrt{n}\|u\|_2 \leq n\|u\|_\infty, \quad \forall u \in \mathbb{R}^n. \tag{1}$$

   (b) Use these inequalities to show that if $A \in \mathbb{R}^{n\times n}$, then the corresponding matrix norms also have equivalence relationships:

   $$\|A\|_1 \leq \sqrt{n}\|A\|_2 \leq n\|A\|_1, \tag{2}$$
   $$\|A\|_\infty \leq \sqrt{n}\|A\|_2 \leq n\|A\|_\infty. \tag{3}$$

   (c) Derive the analogous equivanence relationships for the corresonding condition numbers (assume now that A is inverible).

   (d) Show that for any norm $\|\cdot\|$ on $\mathbb{R}^n$, if $\rho(A)$ is the spectral radius of $A \in \mathbb{R}^{n\times n}$, then

   $$\rho(A) \leq \|A\|.$$

2. (*Gaussian Elimination and LU Factorization*)

   Let $A \in \mathbb{R}^{n\times n}$ be a real, square, $n \times n$ nonsingular matrix, and let $x \in \mathbb{R}^n$ be an n-vector.

   (a) Show that matrix-vector multiplication $y = Ax$ can be expressed as inner-products involving the rows of A.

   (b) Show that matrix-vector multiplication $y = Ax$ can be expressed as a linear combination of the columns of A.

   (c) If we have computed the LU decomposition of A, so that $A = LU$ with L lower-triangular, and U upper-triangular, give the two-step algorithm that produces the solution to the system $Ax = b$, by solving only systems involving L and U.

   (d) Without any assumptions on the nonzero structure of A, what is the complexity of the factorization step, and the complexities of the forward/backsolve steps?

3. (*QR Decomposition and Least Squares*)

   Let $Q \in \mathbb{R}^{n \times n}$ be an *orthogonal matrix*, so that $Q^T = Q^{-1}$.

   (a) Show that if Q is orthogonal, then Q^{-1} is also orthogonal.

   (b) Show that if Q_1 and Q_2 are othogonal, then so is $Q_1 Q_2$.

   (c) Prove that: $(Qx, Qy)_2 = (x, y)_2, \quad \forall x, y \in \mathbb{R}^n$.

   (d) Prove that: $\|Qx\|_2 = \|x\|_2, \quad \forall x \in \mathbb{R}^n$.

   (e) Prove that: $\|Q\|_2 = \|Q^{-1}\|_2 = 1$, so that $\kappa_2(Q) = 1$.

   Let $P \in \mathbb{R}^{n \times n}$ be a projector. Recall that this means P is idempotent ($P^2 = P$). Recall also that to be an *orthogonal projector*, P would also have to be self-adjoint ($P = P^*$), and it would not mean that P is an orthogonal matrix.

   (f) Show that if P is a non-zero projector, then $\|P\|_2 \geq 1$, with equality if and only if P is self-adjoint.

   Consider the overdetermined system: $Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad m \geq n$.

   (g) Show that $A^T A$ is nonsingular if and only if A has full rank.

   Consider finally the particular instance of this overdetermined system:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}. \tag{4}$$

   (h) Before doing any calculation, look at the two equations represented in the system individually, and predict what the least-squares solution will be.

   (i) Calculate a QR decomposition of the coefficient matrix, where Q is a 2×2 rotator (or reflector), and R is 2×1.

   (j) Use the QR decomposition to calculate the least squares solution to (4).

   (k) Instead of using the QR decomposition, form and solve the normal equations to produce the least squares solution to (4).

   (l) Give the general expression for the pseudo-inverse A^+, and then give the particular instance you formed in solving the problem above.

4. (*Eigenvalue Problems*)

   Below we assume $A \in \mathbb{R}^{n \times n}$ unless otherwise noted.

   (a) Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Show that the eigenvalues of A are real, and that eigenvectors corresponding to distinct eigenvalues are real and orthogonal.

   (b) Assume again that $A \in \mathbb{R}^{n \times n}$ is symmetric. State precisely the general existence theorem for the Spectral decomposition of A.

   Consider now the following matrix:

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}.$$

   (c) Starting with the seed vector $q^0 = [1, 1]^T$, perform two steps of the power method with A to get an approximation to the eigenvector corresponding to the largest eigenvalue of A.

   (d) Take your result from part (a) and compute the associated Raleigh quotient, so as to get a corresponding approximation to the largest eigenvalue of A. Of course, we can easily determine the eigenvalues exactly with no computing, due to the form of A. Therefore, comment on how good the approximation is to the largest eigenvalue, after only two steps.

5. (*Conditioning and Stability*)

   Let $A \in \mathbb{C}^{n \times n}$ be nonsingular, and consider the problem:

$$Ax = b, \tag{5}$$

   given A, and also either x or b. Our concern here is the impact on the problem when there is error in one of the three quantities regarded as given as part of the problem specification.

   (a) Show that:

$$\frac{\|\delta x\|}{\|x\|} \le \kappa(A) \frac{\|\delta b\|}{\|b\|},$$

   where $Ax = b$, and where $A(x + \delta x) = (b + \delta b)$. In other words, show that the relative error in the solution to the system $Ax = b$ was bounded by the condition number times the relative error in the data b.

   (b) Consider now also the case that x is given, and we want to calculate b from the product $b = Ax$. Show that if x contains error, i.e. comes to us as $x + \delta x$, then the relative error in b satisfies the *same* inequality:

$$\frac{\|\delta b\|}{\|b\|} \le \kappa(A) \frac{\|\delta x\|}{\|x\|}.$$

   (c) Consider finally the third case, namely that there is error δA in representing A, so that in effect we again solve the wrong system: $(A + \delta A)(x + \delta x) = b$. Show that

$$\frac{\|\delta x\|}{\|x\|} \le \kappa(A) \frac{\|\delta A\|}{\|A\|}.$$

   Let us examine these ideas more generally.

   (d) Give a precise definition of what it means to say that a mathematical problem is "well-posed".

   (e) Can an ill-conditioned problem still be "well-posed"? Explain your reasoning about this.

6. (*Iterative Methods for Linear Systems*)

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix, which as we know means it is also invertible. Assume that A is also very large (e.g. $n \geq 10^6$) and sparse (having $O(1)$ nonzeros per row and column). We would like to solve the linear system $Ax = b$ for $x \in \mathbb{R}^n$, given $b \in \mathbb{R}^n$. For this problem, we will need to make use of the splitting $A = D - L - U$, where D is the diagonal of A, L is the (negative) strictly lower triangle of A, and U is the (negative) strictly upper triangle of A.

(a) Assuming we can find some $B \approx A^{-1}$ and can pick some initial approximation $x^0 \approx x$, show how the Basic Linear Method (BLM):

$$x^{k+1} = (I - \alpha BA)x^k + \alpha Bb, \quad 0 \neq \alpha \in \mathbb{R},$$

arises naturally by deriving some identities that follow from $Ax = b$.

(b) Using Part (a), derive the error propogation equation: $x - x^{k+1} = E(x - x^k)$, where you need to specify what exactly E is.

(c) Using Part (b), derive bounds on the norm of the error at step $k+1$ in terms of the error at step k, and in terms of the error at step $k = 0$.

(d) Use the Banach Fixed-Point Theorem to prove that the BLM converges of α is suitably restricted.

(e) Derive a simplified expression for the error propogator in terms of $\kappa_A(BA)$ when α is chosen to be optimal in terms of maximizing the convergence rate.

(f) Use the splitting of A to specify $B \approx A^{-1}$ for the Jacobi and Gauss-Seidel Methods.

(g) What is the complexity of one iteration of each of the Jacobi and Gauss-Seidel methods for this particular matrix A?

In lecture we derive the Conjugate Gradient method (CG) by starting with the Cayley-Hamilton Theorem, writing the inverse of A as a matrix polynomial in A, and then doing a combination of Gram-Schmidt and orthogonal projection at each step of an iteration. Since A was assumed to be SPD, an alternative mathematically equivalent derivation arises from consider $Ax = b$ as arising as the condition for stationarity of an associated veriational problem.

(h) Show that the linear system $Ax = b$ arises as the necessary condition for minimizing the following energy functional:

$$J(x) = \frac{1}{2}x^T Ax - x^T b.$$

(i) Write down a complete algorithm (usually called the *Steepest Descent* algorithm) as a very simple iteration that steps forward in the direction $p^k = -\nabla J(x^k) = b - Ax^k =: r^k$, starting from an arbitrary initial approximation x^0.

(j) Derive the expression for the step-length α_k that minimizes $J(x^k + \alpha_k p^k)$ at each step k:

$$\alpha_k = \frac{(p^k)^T r^k}{(p^k)^T Ap^k}.$$

(k) Finally, develop a conjugate gradient iteration by modifying the steepest descent algorithm at each iteration, by adding a Gram-Schmidt orthogonalization at each step to force the direction vectors p^k to be mutually A-orthogonal.

(l) Compare the resulting CG iteration to the one we derived in the lectures and the homeworks:

(The Conjugate Gradient Algorithm)
Let $x^0 \in \mathcal{H}$ be given.
$r^0 = b - Ax^0, \quad s^0 = Br^0, \quad p^0 = s^0.$
Do $k = 0, 1, \dots$ until convergence:
$\quad \alpha_k = (r^k, p^k)/(p^k, p^k)_A$
$\quad x^{k+1} = x^k + \alpha_k p^k$
$\quad r^{k+1} = r^k - \alpha_k Ap^k$
$\quad s^{k+1} = Br^{k+1}$
$\quad \beta_{k+1} = -(BAp^k, p^k)_A/(p^k, p^k)_A$
$\quad \gamma_{k+1} = -(BAp^k, p^{k-1})_A/(p^{k-1}, p^{k-1})_A$
$\quad p^{k+1} = BAp^k + \beta_{k+1}p^k + \gamma_{k+1}p^{k-1}$
End do.

(m) *Extra Credit:* Show that these two algorithms are mathematically equivalent.