# Math 210C: Mathematical Physics

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Homework Assignment #3

Due Date: NONE (just some suggested problems to look at to complement the lectures)

**Exercise 3.1.** (*Tensors*) Explain what is mean by the *contraction* of a pair of indices, one covariant and the other contravariant, of a rank $(p, q)$ tensor, to product a rank $(p-1, q-1)$ tensor.

**Exercise 3.2.** (*Differential Forms*) Show that the second-rank tensor given in components by $a_i b_j dx^i \otimes dx^j$ has the same values as $\alpha \otimes \beta$ on any pair of vectors, so that

$$\alpha \otimes \beta = a_i b_j dx^i \otimes dx^j.$$

**Exercise 3.3.** (*Exterior Forms*)

1. Define the tensor project of two covariant rank $p$ and rank $q$ tensors.
2. Define the subset of rank $p$ covariant tensors that are called $p$-forms.
3. Is the tensor project of a $p$-form and a $q$-form a $(p+q)$-form, a is it now only a rank $(p+q)$ covariant tensor? What goes wrong?
4. Define the wedge product of a $pq$ and a $q$ form.
5. Is the wedge project of a $p$-form and a $q$-form now a $(p+q)$-form? Why?

**Exercise 3.4.** (*Exterior Forms*) Show that if $\alpha^p$ is any $p$-form, then it can be expanded as:

$$\alpha^p = \sum_{\underset{\rightarrow}{I}} \alpha^p(\bar{\bar{e}}_I)\sigma^I = \sum_{\underset{\rightarrow}{I}} \alpha(\bar{e}_{i_1}, \dots, \bar{e}_{i_p})\sigma^{i_1} \wedge \dots \wedge \sigma^{i_p},$$

where as in lecture, $\underset{\rightarrow}{I}$ represents increasing index order, and $\bar{\bar{e}}_I$ represents the basis for $E$, and $\sigma^I$ represents the dual basis for $E^*$.

**Exercise 3.5.** (*Exterior Differentiation*) Define exterior differentiation, giving its four main properties, that take as input a $p$-form and produce a $(p+1)$-form.

**Exercise 3.6.** (*Interior Products of p-forms*)

1. Give the definition of the *interior product* of a vector $\bar{v}$ and a $p$-form $\alpha$, producing a $(p-1)$-form $i_{\bar{v}}\alpha$.
2. Explain how this is related to contraction on indices of a general tenesor.

**Exercise 3.7.** (*Anti-derivations*) An *anti-derivation* $f \colon \bigwedge^p \to \bigwedge^{p-1}$ has the following property:

$$f(\alpha^p \wedge \beta^q) = [f\alpha^p] \wedge \beta^q + (-1)^p \alpha^p \wedge [f\beta^q].$$

1. Show that the interior product is an anti-derivation.
2. Show that exterior differentiation is an anti-derivation.

**Exercise 3.8.** Let $\bar{x} = (x, y, z)$ be the orthogonal cartesian coordinate system in $\mathbb{R}^3$. Let $f$ be a 0-form (a function), let $\alpha^1$ be a 1-form associated with a vector $\bar{A}$, let $\gamma^1$ be a 1-form associated with a vector $\bar{C}$, let $\beta^2$ be a 2-form associated with a vector $\bar{B}$ through the relationship:

$$\beta^2 = i_{\bar{B}} \text{ vol},$$

where $i_{\bar{B}}$ is the interior product, and the volume is the standard one in $\mathbb{R}^3$:

$$\text{vol} = \text{vol}^3 = dx \wedge dy \wedge dz.$$

Show that the following relationships between the wedge product, interior product, and exterior differentiation with the standard operations in vector calculus in $\mathbb{R}^3$:

1. $\alpha^1 \wedge \gamma^1 = i_{\bar{A}\times\bar{C}} \text{ vol} \Longleftrightarrow \bar{A} \times \bar{C}$.
2. $\alpha^2 \wedge \beta^2 = \bar{A} \cdot \bar{C} \text{ vol}$.
3. $i_{\bar{C}}\alpha^1 = \bar{C} \cdot \bar{A}$.
4. $i_{\bar{C}}\beta^2 \Longleftrightarrow -\bar{C} \times \bar{B}$.
5. $df \Longleftrightarrow \nabla f$.
6. $d\alpha^1 = i_{\text{curl } \bar{A}} \text{ vol} \Longleftrightarrow \textit{curl } \bar{A}$.
7. $d\beta^2 = \text{div } \bar{B} \text{ vol} \Longleftrightarrow \text{div } \bar{B}$.
8. $di_{\text{grad } f} \text{ vol} = (\nabla^2 f) \text{ vol} \Longleftrightarrow \nabla^2 f$.