Math 210C: Mathematical Physics

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Homework Assignment #2

Due Date: NONE (just some suggested problems to look at to complement the lectures)

Exercise 2.1. (*A Bit of Sets and Topology*)

1. Define the open and closed balls in $\mathbb{R}^n$.
2. Give the definition of a topology that can be placed on a set M.
3. Give the definition of a topological space M.
4. What is the separation property that a Hausdorff topological space has.
5. Give the definition of a metric on a set.
6. Give the definition of a metric space.
7. Argue that a metric space is always a topological space.
8. Argue that a normed space is always a metric space.
9. Argue that an inner-product space is always a normed space.

Exercise 2.2. (*Maps on Toplogical Spaces*) Let $F\colon M \to N$ be a map between two topological spaces M and N. Give definitions of the following properties of F:

1. injection (1-1)
2. surjection (onto)
3. bijection

Why does a map being bijective imply it has an inverse?

Exercise 2.3. (*Maps on Toplogical Spaces*) Let $F\colon M \to N$ be a map between two topological spaces M and N. Give definitions of the following properties of F:

1. isomorphism
2. homeomorphism
3. differential homeomorphism
4. diffeomorphsm

Exercise 2.4. (*General Manifolds*)

1. Explain what is meant by providing a set of charts ("patches" together with corresponding "local coordinate systems") for a set.
2. Give the definition of a differentiable n-manifold.
3. Give the definition of a C^∞ n-manifold.
4. Give the definition of an analytic n-manifold.

Exercise 2.5. (*Vectors, Tangent spaces, Covectors, Cotangent spaces*)

1. Define a (tangent) vector to a submanifold of $\mathbb{R}^n$, and to a general n-manifold M^n.
2. Define a (cotangent) covector to a submanifold of $\mathbb{R}^n$, and to a general n-manifold M^n.
3. Define the tangent space to an n-manifold M^n at the point p.
4. Define the cotangent space to an n-manifold M^n at the point p.

Exercise 2.6. (*Implicitly Defined Submanifolds of* $\mathbb{R}^n$) Investigate the implicitly defined submanifold:

$$M = \{\ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = c\ \},$$

in three distinct cases: $c > 0$, $c < 0$, $c = 0$. Are and/all of the three submanifolds? Do answers change if the origin is excluded? Draw all three level sets (loci) in one picture.

Exercise 2.7. (*Submanifolds of* $\mathbb{R}^n$) Let M^n be a differentiable n-manifold, and let (U, x_U) be a chart (a patch U with a corresonding local coordinate system $x_U = (x_U^1, \ldots, x_U^n)$. Let us try to define a type of "global norm" on vector fields $\bar{x}$ over M^n as:

$$\|\bar{x}\|^2 = \sum_{j=1}^{n} |x_u^i|^2.$$

Is this a "norm" in the sense that we have discussed? What is wrong?

Exercise 2.8. (*Submanifolds of* $\mathbb{R}^n$ *and* M^n) Let M^n be a differentiable submanifold of $\mathbb{R}^N$ that does not contain the origin. Consider now $f\colon M^n \to \mathbb{R}$ defined to be the function that assigns to each point of M^n the square of its distance from the origin. Using local coordinates $(u^1, \ldots, u^n)$, show that a point $p \in M^n$ is a crticial point for this distance function if and only of the position vector to this point is normal to the submanifold M^n.

Exercise 2.9. (*Vector fields and Flows in* $\mathbb{R}^n$ *and* M^n) Consider the vector field on $\mathbb{R}$ defined as $v(x) = x^2(d/dx)$. I.e., $v(x)$ moves x^2 distance in the coordinate direction d/dx. Find the flow $\phi_t(p)$ corresponding to this vector field by solving the differential equation

$$\frac{dx}{dt} = x^2, \qquad , x(0) = p.$$

Now define the open interval containing p as $U_p = (1/2, 3/2)$. Find the largest ϵ so that $\phi\colon U_p \times (-\epsilon, \epsilon) \to \mathbb{R}$ is well-defined. I.e., find the largest t for which the integral curve $\phi_t(p)$ is well-defined for all $p \in U_p$.

Exercise 2.10. (*Covectors in* $\mathbb{R}^n$ *and* M^n) If v is a vector and α is a covector, compute (directly in coordinates) that

$$\sum_{i=1}^{n} a_i^V v_V^i = \sum_{i=1}^{n} a_i^U v_U^i.$$

I.e., you have shown that this quantity is invariant under coordinate transformation.

Exercise 2.11. (*Tensors and Metrics in* $\mathbb{R}^n$ *and* M^n) Let x, y, z be the standard orthogonal cartesian coordinates in $\mathbb{R}^3$, the basis vectors for which we denote as ∂_x, ∂_y, and ∂_z. Let $u^1 = r$, $u^2 = \theta$, $u^3 = \phi$ be spherical coordinates, with corresponding basis vectors ∂_r, ∂_θ, and ∂_ϕ. Recall that the relationship between (x, y, z) and (r, θ, ϕ) is:

$$x = r\,\sin\theta\,\cos\phi,$$
$$y = r\,\sin\theta\,\sin\phi,$$
$$z = r\,\cos\theta.$$

1. Use the chain rule to compute the metric tensor components for spherical coordinates:

$$g_{r\theta} = g_{12} = \langle \partial_r, \partial_\theta \rangle, \ldots$$

2. Confirm that the basis functions ∂_r, ∂_θ, and ∂_ϕ are mutually orthogonal just as as ∂_x, ∂_y, and ∂_z are mutually orthogonal, BUT they are NOT unit length.
3. Compute the coefficients of the gradient with respect to spherical coordinates:

$$\nabla f = (\nabla f)^r \partial_r + (\nabla f)^\theta \partial_\theta + (\nabla f)^\phi \partial_\phi.$$

4. Finally, coefficients of the Laplacean with respect to spherical coordinates: (This one is actually not that easy.)

$$\nabla^2 f = \nabla \cdot (\nabla f).$$

Exercise 2.12. (*2-Tensors and Metrics in* $\mathbb{R}^n$ *and* M^n) Repeat Problem 2.11 but for cylindrical coordinates in $\mathbb{R}^3$.

Exercise 2.13. (*2-Tensors and Metrics in* $\mathbb{R}^n$ *and* M^n) Repeat Problem 2.11 but for polar coordinates in $\mathbb{R}^2$.

Exercise 2.14. (*Tangent and Cotangent Spaces and Bundles*) Let $F\colon M^n \to W^r$ and $G\colon W^r \to V^s$ be smooth maps, where M^n, W^r, V^s are differentiable manifolds. Let (x, y, z) be local coordinates near $p \in M^n$, $F(p) \in W^r$, and $G(F(p)) \in V^s$, respectively, and consider the composite map $G \circ F\colon M^n \to V^s$.

1. Using bases $\partial_x, \partial_y, \partial_z$, show that the differentials obey:

$$(G \circ F)_* = G_* \circ F_*.$$

2. Using bases dx, dy, dz, show that the differentials obey:

$$(G \circ F)^* = G^* \circ F^*.$$

Exercise 2.15. (*General Tensors and Exterior Forms*) Let $A\colon E \to E$ be a linear transformation on a vector space E.

1. Show that the trace $tr(A) = \sum_{i=1}^n A^i_i$ is a true scalar (independent of coordinate transformation) by using the basic transformation properties of mixed tensors. Here A^i_j are the mixed components of A with respect to both the vector space E and the dual space E^*.
2. What about the similar quantity computed with respect to only one of the bases, e.g., $\sum_{i=1}^n A_{ii}$? Is this a scalar?

Exercise 2.16. (*General Tensors and Exterior Forms*) Let $\bar{v} = v^i \partial_i$ (summation convention) be a (contravariant) vector field on a differentiable manifold M^n.

1. Show that $v_j = g_{ji} v^i$ are the components of a corresponding covector representation of $\bar{v}$, by showing that the required transformation properties hold. You will need to use the chain rule:

$$\frac{\partial}{\partial y^i}\left(\frac{\partial y^j}{\partial x^k}\right) = \sum_{r=1}^n \left(\frac{\partial^2 y^j}{\partial x^r \partial x^k}\right)\left(\frac{\partial x^r}{\partial y^i}\right).$$

2. Does $\partial_j v^i$ produce a tensor?
3. Does $(\partial_i v^j - \partial_j v^i)$ produce a tensor?