Following the argument similar to that employed in Example 6.5.6, we write the solution in the form

$$u=\sum_{n=1}^{\infty} a_n(t)\phi_n(x,y)$$

and we assume that

$$f(x,y)=\sum_{n=1}^{\infty} f_n\phi_n(x,y).$$

Consequently, the functions a_n satisfy

$$a_n'(t)=-\lambda_n a_n(t)$$

so that

$$a_n(t)=A_n e^{-\lambda_n t}.$$

We choose $A_n=f_n$ to obtain the solution

$$u(x,y,t)=\sum_{n=1}^{\infty} f_n e^{-\lambda_n t}\phi_n(x,y). \tag{6.5.73}$$

A simple check reveals that (6.5.73) satisfies the differential equation and the boundary and initial conditions.

6.6. Exercises

(1) Let f and g be continuous functions on $\mathbf{R}^N$. Show that if

$$\int_{\mathbf{R}^N} f(x)\phi(x)\,dx=\int_{\mathbf{R}^N} g(x)\phi(x)\,dx$$

for every $\phi\in\mathscr{C}^\infty(\mathbf{R}^N)$ with compact support, then

$$f(x)=g(x) \qquad \text{for every } x\in\mathbf{R}^N.$$

(2) Show that a test function ψ is of the form $\psi(x)=(x\phi(x))'$, where ϕ is a test function, if and only if

$$\int_{-\infty}^{0}\psi(x)\,dx=0 \quad\text{and}\quad \int_0^\infty \psi(x)\,dx=0.$$

(3) Show that $\mathscr{D}$ is a vector space.

(4) Show that if $\phi,\psi\in\mathscr{D}$, then

(a) $f\phi\in\mathscr{D}$ for every smooth function f,
(b) $\{\phi(Ax)\}\in\mathscr{D}$ for every affine transformation A of $\mathbf{R}^N$ onto $\mathbf{R}^N$,
(c) $\phi*\psi\in\mathscr{D}$.

(5) Construct a test function ϕ such that $\phi(x)=1$ for $|x|\le 1$, and $\phi(x)=0$ for $|x|\ge 2$.

(6) Which of the following expressions define a distribution?

(a) $\langle f,\phi\rangle=\sum_{n=1}^{m}\phi^{(n)}(0)$;
(b) $\langle f,\phi\rangle=\sum_{n=1}^{m}\phi(x_n)$, $x_1,\dots,x_m\in\mathbf{R}$ are fixed;
(c) $\langle f,\phi\rangle=\sum_{n=1}^{\infty}\phi^{(n)}(0)$;
(d) $\langle f,\phi\rangle=\sum_{n=1}^{\infty}\phi(x_n)$, $x_1,x_2,\dots\in\mathbf{R}$ are fixed;
(e) $\langle f,\phi\rangle=\sum_{n=1}^{m}\phi^{(n)}(x_n)$, $x_1,\dots,x_m\in\mathbf{R}$ are fixed;
(f) $\langle f,\phi\rangle=(\phi(0))^2$;
(g) $\langle f,\phi\rangle=\sup\phi$;
(h) $\langle f,\phi\rangle=\int_{-\infty}^{\infty}|\phi(t)|\,dt$;
(i) $\langle f,\phi\rangle=\int_a^b\phi(t)\,dt$;
(j) $\langle f,\phi\rangle=\sum_{n=1}^{\infty}\phi(x_n)$, where $\lim_{n\to\infty}x_n=0$.

(7) Let $\phi_n\xrightarrow{\mathscr{D}}\phi$ and $\psi_n\xrightarrow{\mathscr{D}}\psi$. Prove the following:

(a) $a\phi_n+b\psi_n\xrightarrow{\mathscr{D}}a\phi+b\psi$ for any scalars a, b,
(b) $f\phi_n\xrightarrow{\mathscr{D}}f\phi$ for any smooth function f defined on $\mathbf{R}^N$,
(c) $\phi_n\circ A\xrightarrow{\mathscr{D}}\phi\circ A$ for any affine transformation A of $\mathbf{R}^N$ onto $\mathbf{R}^N$,
(d) $D^\alpha\phi_n\xrightarrow{\mathscr{D}}D^\alpha\phi$ for any multi-index α.

(8) Let f be a locally integrable function on $\mathbf{R}^N$. Prove that the functional F on $\mathscr{D}$ defined by

$$\langle F,\phi\rangle=\int_{\mathbf{R}^N}f\phi$$

is a distribution.

(9) Find the nth distributional derivative of $f(x)=|x|$.

(10) Let $f_n(x)=\sin nx$. Show that $f_n\to 0$ in the distributional sense.

(11) Let $\{f_n\}$ be the sequence of functions on $\mathbf{R}$ defined by

$$f_n(x)=\begin{cases}0, & \text{if } x<-1/2n;\\ n, & \text{if } -1/2n\le x\le 1/2n;\\ 0, & \text{if } x>1/2n.\end{cases}$$

Show that the sequence converges to the Dirac delta distribution.

(12) Show that the sequence of Gaussian functions on **R** defined by

$$f_n(x)=\frac{n}{\sqrt{\pi}}e^{-n^2x^2}, \qquad n=1,2,\ldots,$$

converges to the Dirac delta distribution.

(13) Show that the sequence of functions on **R** defined by

$$f_n(x)=\frac{\sin nx}{\pi x}, \qquad n=1,2,\ldots,$$

converges to the Dirac delta distribution.

(14) Let $\phi_0\in\mathscr{D}(\mathbf{R})$ be a fixed test function such that $\int_{-\infty}^{\infty}\phi_0(x)\,dx=1$. Show that every test function $\phi\in\mathscr{D}(\mathbf{R})$ can be represented in the form

$$\phi=K\phi_0+\phi_1,$$

where K is a constant and ϕ_1 is a test function such that $\int_{-\infty}^{\infty}\phi_1(x)\,dx=0$. Moreover, the representation is unique.

(15) The fundamental solution of the one dimensional diffusion equation satisfies the equation

$$G_t-KG_{xx}=\delta(x-\xi)\delta(t-\tau), \qquad -\infty<x<\infty,\ t>0.$$

Show that

$$G(x,t;\xi,\tau)=\frac{H(t-\tau)}{\sqrt{4K\pi(t-\tau)}}\exp\left[-\frac{(x-\xi)^2}{4K(t-\tau)}\right].$$

Hence obtain the solution of the non-homogeneous equation

$$u_t-Ku_{xx}=f(x,t), \qquad -\infty<x<\infty,\ t>0.$$

(16) Find the fundamental solution for the one dimensional diffusion equation

$$u_t-Ku_{xx}=0, \qquad -\infty<x<\infty,\ t>0.$$

(17) Apply the joint Fourier and Laplace transforms to obtain the Green's function for the wave equation

$$G_{tt}-c^2G_{xx}=\delta(x)\delta(t), \qquad -\infty<x<\infty,\ t>0,$$
$$G(x,0)=G_t(x,0)=0.$$

(18) (a) Show that the fundamental solution $G(x,\xi,t)$ for the Cauchy problem

$$G_{tt}=c^2G_{xx}, \qquad -\infty<x<\infty,\ t>0,$$
$$G(x,0)=0, \qquad G_t(x,0)=\delta(x-\xi),$$

is

$$G(x,\xi,t)=\frac{1}{2c}[H(x-\xi+ct)-H(x-\xi-ct)].$$

(b) Use this fundamental solution to solve a more general wave problem

$$u_{tt}=c^2u_{xx}, \qquad -\infty<x<\infty,\ t>0,$$
$$u(x,0)=0, \qquad u_t(x,0)=g(x).$$

(19) Prove the existence of the weak solution of the Dirichlet boundary value problem

$$-\nabla^2u+cu=f \quad \text{in } \Omega\subset\mathbf{R}^2, \qquad u=0 \quad \text{on } \partial\Omega,$$

where c is a positive function of x and y. Show that the weak solution is given by

$$\int_\Omega v(-\nabla^2u+cu)\,d\tau=\int_\Omega fv\,d\tau,$$

where $u,v\in H_0^1(\Omega)$.

(20) Show that the Dirichlet problem for the biharmonic operator

$$\Delta^2u=f \quad \text{in } \Omega, \qquad f\in L^2(\Omega),$$
$$u=\frac{\partial u}{\partial n}=0 \qquad \text{on } \partial\Omega,$$

where $\Omega\subset\mathbf{R}^N$, has a weak solution $u\in H_0^2(\Omega)$ given by

$$\int_\Omega \Delta u\,\Delta v\,d\tau=\int_\Omega fv\,d\tau \qquad \text{for every } v\in H_0^2(\Omega).$$

(21) Show that the boundary value problem

$$-\Delta u+u=f \quad \text{in } \mathbf{R}^N, \qquad f\in L^2(\mathbf{R}^N),$$
$$u\to 0 \quad \text{as } |x|\to\infty$$

has a unique solution $u\in H^1(\mathbf{R}^N)$ such that

$$\int_{\mathbf{R}^N}\nabla u\cdot\nabla v\,d\tau+\int_{\mathbf{R}^N}uv\,d\tau=\int_{\mathbf{R}^N}fv\,d\tau$$

for all $v\in H^1(\mathbf{R}^N)$.

(22) Let $\Omega \subset \mathbf{R}^N$ be a bounded open set. Consider the Robin boundary value problem

$$-\Delta u + u = f \quad \text{in } \Omega, \qquad f \in L^2(\Omega),$$
$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \partial\Omega, \qquad \alpha > 0.$$

Show that there exists a unique solution $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \qquad \text{for every } v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\tau + \int_\Omega uv \, d\tau + \alpha \int_{\partial\Omega} uv \, d\tau, \qquad u, v \in H_0^1(\Omega).$$

(23) Use the Fourier transform method to show that the solution of the telegrapher's problem

$$u_{tt} + au_t + bu = c^2 u_{xx}, \qquad -\infty < x,\ t < \infty,$$
$$u(0, t) = f(t), \qquad u_x(0, t) = g(t),$$

is

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\tilde{f}(k) \cos\{x\alpha(k)\} + \frac{\tilde{g}(k)}{\alpha(k)} \sin\{x\alpha(k)\} \right] e^{ikx} \, dk,$$

where

$$\alpha(k) = \frac{b + ika - k^2}{c^2},$$

and $\tilde{f}$ and $\tilde{g}$ are the Fourier transforms of f and g, respectively.

(24) Find the solution of the Dirichlet problem in the half-plane

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty,\ y > 0,$$
$$u(x, 0) = f(x), \quad -\infty < x < \infty,$$
$$u \text{ is bounded as } y \to \infty, \qquad u, u_x \text{ vanish as } |x| \to \infty.$$

(25) Find the solution of the Neumann problem in the half-plane

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty,\ y > 0,$$
$$u_y(x, 0) = g(x), \qquad -\infty < x < \infty,$$
$$u \text{ is bounded as } y \to \infty, \qquad u, u_x \text{ vanish as } |x| \to \infty.$$

(26) Find the solution of the system

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty,\ 0 \le y \le a,$$
$$u(x, 0) = f(x), \qquad u(x, a) = g(x).$$

(27) Solve the boundary value problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty,\ 0 \le y \le a,$$
$$u(x, 0) = f(x), \qquad u_y(x, a) = 0.$$

(28) Show that the solution of the slow motion of viscous fluid through a slit governed by the biharmonic equation (6.5.53) with the boundary conditions

$$\psi_y = g(x) H(a - |x|), \qquad \psi_x = 0 \quad \text{on } y = 0$$

is

$$\psi(x, y) = \frac{y^2}{\pi} \int_{-\infty}^{\infty} \frac{g(x') \, dx'}{(x - x')^2 + y^2}.$$

(29) Use the analysis of Example 6.5.4 to find the solution of the two dimensional steady flow of an inviscid liquid through a slit in a plane rigid boundary $y = 0$. The problem is to find the velocity potential $\phi(x, y)$ satisfying the Laplace equation with the boundary conditions

$$\phi = H(a - |x|), \qquad v = -\phi_y = (a^2 - x^2)^{1/2} H(a - |x|) \quad \text{on } y = 0.$$

(30) If $E(u, v)$ is a bilinear form defined by the Dirichlet integral (6.3.36) of a self-adjoint operator L, prove that

(a) $E(\alpha u + \beta v, \alpha u + \beta v) = \alpha^2 E(u, v) + 2\alpha\beta E(u, v) + \beta^2 E(v, v)$, where α and β are constants;
(b) $(E(u, v))^2 \le E(u, u) E(v, v)$, if $b \le 0$ and L is an elliptic operator.