Example 5.11.7. Consider the integral equation

$$\int_{-\infty}^{\infty} e^{-|x-t|}u(t)\,dt = -\frac{1}{4}u(x) + e^{-|x|}, \qquad -\infty < x < \infty. \tag{5.11.32}$$

Application of the Fourier transform with respect to x yields

$$\frac{2}{1+k^2}\hat{u}(k) = -\frac{1}{4}\hat{u}(k) + \sqrt{\frac{2}{\pi}}\frac{1}{1+k^2},$$

so that

$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}}\frac{8}{k^2+9}.$$

By the inverse Fourier transform, we obtain

$$u(x) = \frac{4}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^2+9}\,dk.$$

To evaluate this integral for $x > 0$ we use a semicircular closed contour in the lower half of the complex plane. It turns out that

$$u(x) = \frac{4}{3}e^{-3x}, \qquad x > 0.$$

Similarly, for $x < 0$, we use a closed semicircular contour in the upper half of the complex plane to obtain

$$u(x) = \frac{4}{3}e^{3x}, \qquad x < 0.$$

Hence the solution of (5.11.32) is

$$u(x) = \frac{4}{3}e^{-3|x|}. \tag{5.11.33}$$

5.12. Exercises

(1) Determine the fixed points, if any, of the following operators:

(a) $T(x) = x + a$ on any vector space;
(b) $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x, 0)$;
(c) $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (y, y)$;
(d) $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x, y) = (x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi),$$

where ϕ is a fixed real number.

(2) Suppose T is an operator on $\mathscr{C}([0, 1])$ defined by

$$(Tu)(t) = \int_0^t (u(x))^2\,dx.$$

Show that T is not a contraction on the closed unit ball in $\mathscr{C}([0, 1])$, but that it is one on the closed ball of radius $\frac{1}{4}$ in $\mathscr{C}([0, 1])$.

(3) Show that the operator $T: \mathscr{C}([0, 1]) \to \mathscr{C}([0, 1])$ defined by

$$(Tx)(t) = x(0) + \lambda\int_0^t x(\tau)\,d\tau, \qquad \lambda \in \mathbf{R},$$

is a contraction provided $|\lambda| < 1$.

(4) Show that the non-linear integral equation

$$f(x) = \int_0^1 e^{-sx}\cos(\alpha f(s))\,ds, \qquad 0 \le x \le 1, 0 < \alpha < 1,$$

has a unique solution.

(5) Consider a system of ordinary differential equations

$$\frac{d}{dx}\phi_k(x) = f_k(x_0, \phi_1(x), \phi_2(x), \ldots, \phi_N(x))$$

with the initial data

$$\phi_k(x_0) = y_{0k},$$

where $k = 1, 2, \ldots, N$; and the functions $f_k(x_0, y_1, y_2, \ldots, y_N)$ are continuous in some domain $\Omega \subset \mathbf{R}^{N+1}$, and $(x_0, y_{01}, y_{02}, \ldots, y_{0N}) \in \Omega$. Moreover, we assume that the functions f_k satisfy the Lipschitz condition

$$|f_k(x, y_1, y_2, \ldots, y_N) - f_k(x, z_1, z_2, \ldots, z_N)| \le \sup_x \max_{1\le m\le N} |y_m - z_m|$$

in Ω. Prove that this system has a unique system of solutions $y_k = \phi_k(x)$ in some interval $|x - x_0| < d$.

(6) Use the method presented in Section 5.6 to solve the following homogeneous Fredholm equation:

$$f(x) = \lambda\int_{-1}^{1}(x+t)f(t)\,dt.$$

(7) Use the method presented in Section 5.6 to solve the following non-homogeneous equation:

$$f(x) = \phi(x) + \lambda\int_0^1 (\pi x\sin\pi t + 2\pi x^2\sin 2\pi t)f(t)\,dt.$$

(8) Express the solution of the integral equation

$$f(x)=\phi(x)+\lambda\int_0^{2\pi}\cos(x+t)f(t)\,dt$$

in the resolvent form

$$f(x)=\phi(x)+\lambda\int_0^{2\pi}\Gamma(x,t;\lambda)\phi(t)\,dt,$$

where λ is not an eigenvalue. Obtain the general solution, if it exists, for $\phi(x)=\sin x$.

(9) Show that the solution of the differential equation

$$\frac{d^2f}{dx^2}+xf=1,\qquad f(0)=f'(0)=0,$$

satisfies the non-homogeneous Volterra equation

$$f(x)=\frac{1}{2\pi^2}+\int_0^x t(t-x)f(t)\,dt.$$

(10) Transform the problems

(a) $\dfrac{d^2f}{dx^2}+f=x,\qquad f(0)=0,\ f'(1)=0,$

(b) $\dfrac{d^2f}{dx^2}+f=x,\qquad f(0)=1,\ f'(1)=0,$

into Fredholm integral equations.

(11) Discuss the solutions of the integral equation

$$f(x)=\phi(x)+\lambda\int_0^1(x+t)f(t)\,dt.$$

(12) When do the following integral equations have solutions?

(a) $f(x)=\phi(x)+\lambda\int_0^1(1-3xt)f(t)\,dt.$

(b) $f(x)=\phi(x)+\lambda\int_0^{2\pi}\sin(x+t)f(t)\,dt.$

(c) $f(x)=\phi(x)+\lambda\int_0^1 xtf(t)\,dt.$

(d) $f(x)=\phi(x)+\lambda\int_{-1}^1\sum_{n=1}^{m}P_n(x)P_n(t)f(t)\,dt,$

where P_n is the nth degree Legendre polynomial.

(e) $f(x)=x+\frac{1}{2}\int_{-1}^1(x+t)f(t)\,dt.$

(13) Find the eigenvalues and eigenfunctions of the following integral equations:

(a) $f(x)=\lambda\int_0^{2\pi}\cos(x-t)f(t)\,dt.$

(b) $f(x)=\lambda\int_{-1}^1(t-x)f(t)\,dt.$

(c) $f(x)=\phi(x)+\lambda\int_0^{2\pi}\cos(x+t)f(t)\,dt.$

(14) Solve the integral equations

(a) $f(x)=\phi(x)+\lambda\int_0^1 tf(t)\,dt.$

(b) $f(x)=x+\lambda\int_0^{1/2}f(t)\,dt.$

(c) $f(x)=\frac{5x}{6}+\frac{1}{2}\int_0^1 xtf(t)\,dt.$

(d) $f(x)=x+\int_0^1(1+xt)f(t)\,dt.$

(e) $f(x)=e^x+\lambda\int_0^1 2e^{x+t}f(t)\,dt.$

(15) Use the separable kernel method to show that

$$f(x)=\lambda\int_0^1\cos x\sin t\,f(t)\,dt$$

has no solution except the trivial solution $f=0$.

(16) Obtain the Neumann n series solutions of the following equations:

(a) $f(x)=x+\frac{1}{2}\int_{-1}^1(t+x)f(t)\,dt.$

(b) $f(x)=x+\int_0^x(t-x)f(t)\,dt.$

(c) $f(x)=x-\int_0^x(t-x)f(t)\,dt.$

(d) $f(x)=1-2\int_0^x tf(t)\,dt.$

(17) If $Lu = u'' + \omega^2 u$, show that L is formally self-adjoint and the concomitant is $J(u, v) = vu' - uv'$. Moreover, if u is a solution of $Lu = 0$ and v is a solution of $L^* v = 0$, then the concomitant of u and v is a constant.

(18) Let L be a self-adjoint differential operator given by (5.8.15). If u_1 and u_2 are two solutions of $Lu = 0$, and $J(u_1, u_2) = 0$ for some x for which $a_2(x) \neq 0$, then u_1 and u_2 are linearly independent.

(19) Consider the differential operator

$$L = e^x D^2 + e^x D, \qquad D = \frac{d}{dx}, \quad 0 \le x \le 1,$$

$$u'(0) = 0, \qquad u(1) = 0.$$

Show that L is formally self-adjoint.

(20) Prove continuity of the Green's function defined in Theorem 5.10.1.

(21) Find eigenvalues and eigenfunctions of the following Sturm–Liouville system:

$$u'' + \lambda u = 0, \qquad 0 \le x \le \pi,$$

$$u(0) = u'(\pi) = 0.$$

(22) Transform the Euler equation

$$x^2 u'' + x u' + \lambda u = 0, \qquad 1 \le x \le e,$$

with the boundary conditions

$$u(1) = u(e) = 0$$

into the Sturm–Liouville system

$$\frac{d}{dx}\left[x \frac{du}{dx}\right] + \frac{1}{x} \lambda u = 0,$$

$$u(1) = u(e) = 0.$$

Find the eigenvalues and eigenfunctions.

(23) Prove that $\lambda = 0$ is not an eigenvalue of the system defined in Example 5.9.1.

(24) Show that the Sturm–Liouville operator $L = DpD + q$, $D = d/dx$ is positive if $p(x) > 0$ and $q(x) \ge 0$ for all $x \in [a, b]$.

(25) Show that the Sturm–Liouville operator L in $L^2([a, b])$ given by

$$L = \frac{1}{r(x)} (DpD + q)$$

is not symmetric.

(26) Use the Fourier transform to solve the forced linear harmonic oscillator

$$\ddot{x} + \omega^2 x = a \sin \Omega t, \qquad t > 0, \omega \neq \Omega, \qquad x(0+) = 0 = \dot{x}(0+).$$

Examine the case when $\omega = \Omega$.

(27) Solve the problem discussed in Example 5.11.1 with $E(t) = E_0 e^{-\alpha t} \sin \omega t \, H(t)$ and $I(0+) = I_0$.

(28) If there is a capacitor in the circuit discussed in Example 5.11.1, then the current $I(t)$ satisfies the following integrodifferential equation:

$$L \frac{dI}{dt} + RI + \frac{1}{C}\left[q_0 + \int_0^t I(t)\, dt\right] = E(t),$$

where q_0 is the initial charge on the capacitor so that

$$q = q_0 + \int_0^t I(t)\, dt$$

is the charge and $dq/dt = I$.

Solve this problem using the Fourier transform and the following conditions

$$I = q = E = 0 \qquad \text{for } t < 0,$$

$$I(0+) = I_0 \text{ and } q(0+) = q_0.$$

Examine the special case when $E(t) = H(t)$.

(29) Use the Fourier transform to solve the following problem:

$$y'' + 3y' + 2y = e^{-x}, x > 0, \qquad y(0+) = y_0 \text{ and } y'(0+) = y_{00}.$$

(30) Use the Fourier transform to solve the following pair of coupled differential systems for $t > 0$:

$$x' + y' - x + 3y = e^{-t},$$

$$x' + y' + 2x + y = e^{-2t},$$

$$x(0+) = x_0 \quad \text{and} \quad y(0+) = y_0.$$

(31) (a) Show that the solution of the integral equation

$$u(x)-\lambda\int_{-\infty}^{\infty} e^{-|x-t|}\,u(t)\,dt=e^{-|x|}$$

is

$$u(x)=\frac{e^{-\sqrt{1-2\lambda}\,|x|}}{\sqrt{1-2\lambda}}\qquad \text{for } \lambda<\frac{1}{2}.$$

(b) If

$$Tu=\int_{-\infty}^{\infty} e^{-|x-t|}\,u(t)\,dt$$

show that $\|T\|\le 2$; $\|\cdot\|$ denotes the norm in $L^2(\mathbf{R})$.

(32) Prove the following properties of the Hilbert transform $\tilde{\phi}(x)=\mathscr{H}\{\phi(t);\,x\}$.

(a) $\mathscr{H}\{\phi(t+a);\,x\}=\mathscr{H}\{\phi(t);\,x+a\}$;
(b) $\mathscr{H}\{\phi(at);\,x\}=\mathscr{H}\{\phi(t);\,ax\}$, $a>0$;
(c) $\mathscr{H}\{\phi(-t);\,x\}=-\mathscr{H}\{\phi(t);\,-x\}$;
(d) $\mathscr{H}\{\phi'(t);\,x\}=(d/dx)\tilde{\phi}(x)$;
(e) $\mathscr{H}\{t\phi(t);\,x\}=x\tilde{\phi}(x)+(1/\pi)\int_{-\infty}^{\infty}\phi(t)\,dt$.

(33) Show that if $f\in L^2(\mathbf{R})$ then $\mathscr{H}\{f\}\in L^2(\mathbf{R})$ and $\|\mathscr{H}\{f\}\|=\|f\|$.

(34) Show that $\mathscr{F}\{\mathscr{H}\{f\}\}=(-i\,\operatorname{sgn} k)\mathscr{F}\{f\}$.

CHAPTER 6

Generalized Functions and Partial Differential Equations

6.1. Introduction

In this chapter, we shall first discuss briefly the basic concepts and properties of distributions. The theory of distributions was initiated by the Russian mathematician S. L. Sobolev in 1936. The concept of distributions was independently introduced by the French mathematician L. Schwartz in the 1950s. Since Schwartz was the one who developed the theory almost to its present form, distributions are often called Schwartz distributions. Distributions have found applications in many areas of mathematics, including differential and integral equations.

The rest of this chapter deals with Green's functions for partial differential equations of most common interest. This is followed by the form of the Green's identity associated with partial differential operators. Section 6.4 discusses weak solutions of elliptic boundary value problems. The final section is devoted to applications of the Fourier transform to partial differential equations of physical interest.

6.2. Distributions

Consider a partial differential operator L of order m in N variables