Since, by Theorem 2.4.2, $\lim_{t\to\infty}\int|f(t-y)-f(-y)|\,dy=0$, the proof is complete.

2.16. Exercises

(1) Denote by $\mathscr{A}$ the family consisting of all finite unions of semi-open intervals $[a, b)$ and the empty set. Prove the following properties of $\mathscr{A}$:

(a) If $A_1, \ldots, A_n \in \mathscr{A}$, then $A_1 \cup \cdots \cup A_n \in \mathscr{A}$.
(b) If $A_1, \ldots, A_n \in \mathscr{A}$, then $A_1 \cap \cdots \cap A_n \in \mathscr{A}$.
(c) If $A, B \in \mathscr{A}$, then $A\backslash B \in \mathscr{A}$.

(2) Show that step functions form a vector space.

(3) Prove that the integral of a step function is independent of a particular representation (2.2.1).

(4) Show that for any step functions f and g we have

(a) $\operatorname{supp}(f+g) \subseteq \operatorname{supp} f \cup \operatorname{supp} g$,
(b) $\operatorname{supp} fg = \operatorname{supp} f \cap \operatorname{supp} g$,
(c) $\operatorname{supp}|f| = \operatorname{supp} f$,
(d) $\operatorname{supp} \lambda f = \operatorname{supp} f$, $\lambda \in \mathbf{R}$, $\lambda \neq 0$,
(e) If $|f| \leq |g|$, then $\operatorname{supp} f \subseteq \operatorname{supp} g$.

(5) Prove Lemma 2.2.1.

(6) Prove Theorem 2.4.2 for step functions.

(7) Expand the following functions into a series of step functions (i.e., find step functions $f_1, f_2, \ldots$ such that $f \simeq f_1 + f_2 + \cdots$):

(a) $$f(x)=\begin{cases}1 & \text{if } x=0,\\ 0 & \text{if } x\neq 0.\end{cases}$$

(b) $$f(x)=\begin{cases}1 & \text{if } x\in[a, b],\\ 0 & \text{if } x\notin[a, b],\end{cases} \qquad a<b.$$

(c) $f(x)=\max\{0, 1-|x|\}$.
(d) f is a piecewise continuous function with bounded support.

(8) Show that $f \in L^1(\mathbf{R})$ if any only if there are intervals $[a_1, b_1), [a_2, b_2), \ldots$ and numbers $\lambda_1, \lambda_2, \ldots$ such that

$$f \simeq \lambda_1 \chi_{[a_1, b_1)} + \lambda_2 \chi_{[a_2, b_2)} + \cdots$$

(9) Show that if $f \simeq f_1 + f_2 + \cdots$ then $f + g \simeq g + f_1 + f_2 + \cdots$ for any step function g. In particular, if $f \simeq f_1 + f_2 + \cdots$ then

$$f - f_1 - \cdots - f_n \simeq f_{n+1} + f_{n+2} + \cdots$$

(10) If $f \in L^1(\mathbf{R})$ and f vanishes outside of a bounded interval J, then there are step functions $f_1, f_2, \ldots$ vanishing outside of J such that $f \simeq f_1 + f_2 + \cdots$.

(11) If $f \in L^1(\mathbf{R})$ and $f \geq 0$, is it always possible to find non-negative step functions $f_1, f_2, \ldots$ such that $f \simeq f_1 + f_2 + \cdots$?

(12) Show that the characteristic function of the set of all rational numbers is Lebesgue integrable but not Riemann integrable.

(13) Define $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$. Prove that $f \in L^1(\mathbf{R})$ if any only if $f^+ \in L^1(\mathbf{R})$ and $f^- \in L^1(\mathbf{R})$.

(14) Show that if f is a continuous integrable function, then there are step functions $f_1, f_2, \ldots$ such that $f \simeq f_1 + f_2 + \cdots$ and $|f| \simeq |f_1| + |f_2| + \cdots$.

(15) By a *tent function* we mean a function of the form

$$f(x)=\begin{cases}2(x-a)/(b-a) & \text{if } a\leq x\leq (a+b)/2,\\ 2(b-x)/(b-a) & \text{if } (a+b)/2\leq x\leq b,\\ 0 & \text{otherwise,}\end{cases}$$

where $a<b$. Show that $f \in L^1(\mathbf{R})$ if and only if there exist tent functions $f_1, f_2, \ldots$ and numbers $\lambda_1, \lambda_2, \ldots$ such that $f \simeq \lambda_1 f_1 + \lambda_2 f_2 + \cdots$

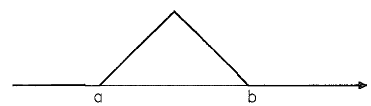

FIGURE 2.3.

(16) Show that the relation

$$f \sim g \quad \text{if} \quad \int |f-g| = 0$$

is an equivalence in $L^1(\mathbf{R})$.

(17) Prove the following:

(a) Every countable subset of $\mathbf{R}$ is a null set.
(b) A countable union of null sets is a null set.

(18) Prove the following properties of convergence almost everywhere:

(a) If $f_n \to f$ a.e. and $\lambda \in \mathbf{R}$, then $\lambda f_n \to \lambda f$ a.e..
(b) If $f_n \to f$ a.e. and $g_n \to g$ a.e., then $f_n + g_n \to f + g$ a.e..
(c) If $f_n \to f$ a.e., then $|f_n| \to |f|$ a.e..

(19) Show that every Lebesgue integrable function can be approximated in norm and almost everywhere by a sequence of continuous functions.

(20) Let $f \in L^1(\mathbf{R})$. Define

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f_n \to f$ i.n..

(21) Show that there exists an unbounded continuous function $f \in L^1(\mathbf{R})$.

(22) Show that if f is a uniformly continuous function on $\mathbf{R}$ and $f \in L^1(\mathbf{R})$, then f is bounded and $\lim_{|x|\to\infty} f(x) = 0$.

(23) Show that locally integrable functions form a vector space.

(24) Let $f \in L^1(\mathbf{R})$ and let g be a bounded locally integrable function. Show that $fg \in L^1(\mathbf{R})$ and $\int |fg| \le \sup_{x\in\mathbf{R}} |g(x)| \int |f|$.

(25) Show that the space $L^1(J)$ is complete for any interval $J \subseteq \mathbf{R}$.

(26) Prove: If a sequence of locally integrable functions $\{f_n\}$ converges almost everywhere to a function f and $|f_n| \le h$ for every $n \in \mathbf{N}$, where h is a locally integrable function, then f is locally integrable.

(27) In Example 2.7.2 we define a sequence of functions $\{f_n\}$ convergent to 0 in norm but divergent at every point of $\mathbf{R}$. Find a subsequence of $\{f_n\}$ convergent to 0 almost everywhere.

(28) Prove: If $\{f_n\}$ is a sequence of integrable functions which is non-decreasing almost everywhere and $|\int f_n| \le M$ for some constant M and all $n \in \mathrm{N}$, then there exists an integrable function f such that $f_n \to f$ i.n. and $f_n \to f$ a.e.. Moreover, we have $|\int f| \le M$.

(29) Prove: If a sequence of integrable functions $\{f_n\}$ converges almost everywhere to a function f and $|f_n(x)| \le h(x)$ for almost all $x \in \mathbf{R}$, all $n \in \mathbf{N}$, and some integrable function h, then f is integrable and $f_n \to f$ i.n..

(30) Show that the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \ne 0, \\ 1 & \text{if } x = 0, \end{cases}$$

is not Lebesgue integrable, although the improper Riemann integral $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

(31) Let $\mathcal{M}$ denote the collection of all measurable subsets of $\mathbf{R}$. Prove the following:

(a) $\varnothing, \mathbf{R} \in \mathcal{M}$.
(b) If $A_1, A_2, \ldots \in \mathcal{M}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.
(c) If $A_1, A_2, \ldots \in \mathcal{M}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.
(d) If $A, B \in \mathcal{M}$, then $A \backslash B \in \mathcal{M}$.
(e) Intervals are measurable sets.
(f) Open subsets of $\mathbf{R}$ are measurable.
(g) Closed subsets of $\mathbf{R}$ are measurable.

(32) Let $\mathcal{M}$ be the collection of all measurable subsets of $\mathbf{R}$ and let μ be the Lebesgue measure on $\mathbf{R}$. Prove the following:

(a) If $A_1, A_2, \ldots \in \mathcal{M}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$.
(b) If $A, B \in \mathcal{M}$ and $A \subseteq B$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$.
(c) If $A_1, A_2, \ldots \in \mathcal{M}$ and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

(33) Let f be a real valued function on $\mathbf{R}$. Show that the following conditions are equivalent:

(a) f is measurable;
(b) $\{x \in \mathbf{R}: f(x) \le \alpha\}$ is measurable for all $\alpha \in \mathbf{R}$.
(c) $\{x \in \mathbf{R}: f(x) < \alpha\}$ is measurable for all $\alpha \in \mathbf{R}$.
(d) $\{x \in \mathbf{R}: f(x) \ge \alpha\}$ is measurable for all $\alpha \in \mathbf{R}$.
(e) $\{x \in \mathbf{R}: f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbf{R}$.

(34) Prove: Let $A_1, A_2, \ldots$ be measurable sets such that $\lim_{n\to\infty} \mu(A_n) = 0$. Then for every $f \in L^1(\mathbf{R})$ we have

$$\lim_{n\to\infty} \int_{A_n} f = 0.$$

(35) Let

$$g(x)=\begin{cases}1/\sqrt{x} & \text{for } 0<|x|<1,\\ 0 & \text{otherwise.}\end{cases}$$

Show that $g \in L^1(\mathbf{R})$ but $g^2 \notin L^1(\mathbf{R})$.

(36) Let $f(x)=\min\{1, 1/|x|\}$. Show that $f \notin L^1(\mathbf{R})$ but $f \in L^2(\mathbf{R})$.

(37) Show that $L^2([a, b]) \subseteq L^1([a, b])$ for any bounded interval $[a, b]$.

(38) Let $f, g, h \in L^1(\mathbf{R})$. Show that $(f+g)*h=f*h+f*h$.

(39) Let f be the characteristic function of the interval $[-1, 1]$. Calculate the convolutions $f*f$ and $f*f*f$.

(40) Let $f \in L^1(\mathbf{R})$ and let g be a bounded continuously differentiable function on $\mathbf{R}$. Show that $f*g$ is differentiable. If, in addition, g' is bounded show that $(f*g)'=f*g'$.

(41) Let f be a locally integrable function on $\mathbf{R}$ and let g be a continuously differentiable function with bounded support in $\mathbf{R}$. Show that $f*g$ is differentiable and $(f*g)'=f*g'$.

CHAPTER 3

Hilbert Spaces and Orthonormal Systems

3.1. Introduction

The theory of Hilbert spaces was initiated by David Hilbert (1862–1943) in his 1912 work on quadratic forms in infinitely many variables which he applied to the theory of integral equations. After many years John von Neumann (1903–1957) first formulated an axiomatic theory of Hilbert spaces and developed the modern theory of operators on Hilbert spaces. His remarkable contribution to this area has provided the mathematical foundation of quantum mechanics. Von Neumann's work has also provided an almost definite physical interpretation of quantum mechanics in terms of abstract relations in an infinite dimensional Hilbert space.

This chapter is concerned with inner product spaces (called also pre-Hilbert spaces) and Hilbert spaces. The basic ideas and properties will be discussed with special attention given to orthonormal systems. The theory is illustrated by numerous examples.