One of the simple nonlinear partial differential equations which exhibits the transition phenomena shown in Figure 8.3 is

$$u_t = \nabla^2 u + \lambda u + u^3 \qquad \text{in } D, \tag{8.9.20}$$

$$u = 0 \qquad \text{on } \partial D, \tag{8.9.21}$$

where D is a smooth bounded domain in $\mathbf{R}^N$. The equilibrium states of (8.9.20) are given by solutions of the time-independent equation ($u_t \equiv 0$). One solution is obviously $u = 0$, which is valid for all λ; this solution becomes unstable at $\lambda = \lambda_1$, the first eigenvalue of the Laplacian on D: $\nabla^2 u_1 + \lambda_1 u_1 = 0$, $u_1 = 0$ on ∂D. For $\lambda > \lambda_1$, there are at least three solutions of the nonlinear equilibrium equation. The nature of the solution set in the neighborhood of $(\lambda_1, 0)$ is given in Figure 8.3; the new bifurcating solutions are stable. The Laplacian has a set of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ which tend to infinity, and all of these eigenvalues are potential bifurcation points.

In the theory of Calculus in Banach spaces, the following version of the Implicit Function Theorem is concerned with the existence, uniqueness and smoothness properties of the solution of the Equation (8.9.1).

Theorem 8.9.1 (Implicit Function Theorem). *Suppose Λ, E, B are real Banach spaces and F is a Fréchet differentiable mapping from a domain $D \subset \Lambda \times E$ to B. Assume $F(\lambda_0, u_0) = 0$ and the Fréchet derivative $F'(\lambda_0, u_0)$ is an isomorphism from E to B. Then, locally, for $\|\lambda - \lambda_0\|$ sufficiently small, there is a differentiable mapping $u(\lambda)$ from Λ to E, with $(\lambda, u(\lambda)) \in D$, such that $F(\lambda, u(\lambda)) = 0$. Moreover, $(\lambda, u(\lambda))$ is the only solution of $F = 0$ in a sufficiently small neighborhood $D' \subset D$. If F is C^n then u is C^n. If Λ, E, and B are complex Banach spaces and F is Fréchet differentiable, then F is analytic and u is analytic in λ.*

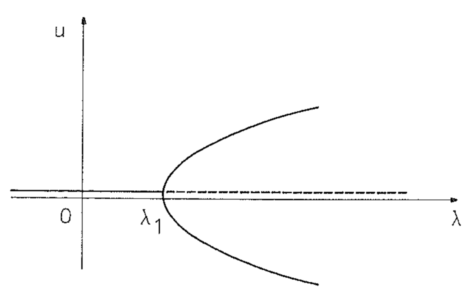

FIGURE 8.3. Bifurcation diagram where unstable solutions are represented by dashed lines.

The proof of the theorem is beyond the scope of this book. However, the theorem can be proved by using a contraction mapping argument and is adequate for most physical applications. The reader is referred to Sattinger (1973) and Dieudonné (1969) for a detailed discussion of proofs.

Bifurcation phenomena typically accompany the transition to instability when a characteristic parameter crosses a critical value, and hence they play an important role in applications to mechanics. Indeed the area of mechanics is a rich source of bifurcation and instability phenomena, and the subject has always stimulated the rapid development of functional analysis.

8.10. Exercises

(1) Let H_1 and H_2 be real Hilbert spaces. Show that if T is a bounded linear operator from H_1 into H_2, and f is a real functional on H_1 defined by

$$f(x) = \|u - Tx\|^2,$$

where u is a fixed vector in H_2, then f has a Fréchet derivative at every point given by

$$f'(x) = -2T^*u + 2T^*Tx,$$

where T^* is the adjoint of T.

(2) Suppose $T: B_1 \to B_2$ is Fréchet differentiable on a open set $\Omega \subset B_1$. Show that if $x \in \Omega$ and $h \in B_1$ are such that $x + th \in \Omega$ for every $t \in [0, 1]$, then

$$\|T(x+h) - T(x)\| \le \|h\| \sup_{0<\alpha<1} \|T'(x+\alpha h)\|.$$

(3) Suppose T is a twice Fréchet differentiable on an open domain Ω in a Banach space B_1. Let $x \in \Omega$ and $x + \alpha h \in \Omega$ for every $\alpha \in [0, 1]$. Prove that

$$\|T(x+H) - T(x) - T'(x)h\| \le \tfrac{1}{2}\|h\|^2 \sup_{0<\alpha<1} \|T''(x+\alpha h)\|.$$

(4) Find the extrema of the following functionals:

(a) $I(y) = \int_1^2 ((y')^2 - 2xy)\, dx, \qquad y(1) = 0,\ y(2) = -1;$

(b) $I(y) = \int_0^1 e^{2x}((y')^2 - y^2 - y)\, dx, \qquad y(0) = 0,\ y(1) = \dfrac{1}{e};$

(c) $I(y)=\int_0^{2\pi}((y')^2-y^2)\,dx, \qquad y(0)=1, y(2\pi)=1;$

(d) $I(y)=\int_0^{a}((y')^2-y^2)\,dx, \qquad y(0)=1, y(a)=1, a\neq k\pi.$

(5) Find the extrema of the following functional:

$$I(u(x), v(y))=\int_1^2((u')^2+v^2+(v')^2)\,dx,$$

$$u(1)=1, \quad u(2)=2, \quad v(1)=0, \quad v(2)=1.$$

(6) Determine the extrema of the functional

$$I(y)=\int_a^b\frac{1}{y}(1+(y')^2)^{1/2}\,dx$$

which passes through the given points (a, y_1) and (b, y_2) in the upper half plane.

(7) Find the shortest distance from the point $(1, 1, 1)$ to the sphere

$$x^2+y^2+z^2=1.$$

(8) Solve the following isoperimetric problem: Minimize the functional

$$I(y)=\int_0^1((y')^2-y^2)\,dx,$$

subject to $\int_0^1\sqrt{1+(y')^2}\,dx=\sqrt{2}$, $y(0)=0$, $y(1)=1$.

(9) Find the minimum of the functional

$$I(y)=\int_0^\pi(y')^2(x)\,dx,$$

subject to the condition $\int_0^\pi y^2(x)\,dx=1$, $y(0)=0$, $y(\pi)=0$.

(10) Derive Newton's Second Law of Motion from Hamilton's principle.

(11) Derive the equation of a simple harmonic oscillator in a non-resisting medium from Hamilton's principle.

(12) Find the curve $y=y(x)$, $0\leq x\leq 1$, with $y(0)=0$, $y(1)=0$, and fixed arc length L that has maximal area

$$A=\int_0^1 y(x)\,dx.$$

(13) Show that the potential function $u(x, y)$ which minimizes the functional

$$I(u)=\iint_\Omega\tfrac{1}{2}(u_x^2+u_y^2)\,dx\,dy, \qquad \Omega\subset\mathbf{R}^2,$$

satisfies the Laplace equation $u_{xx}+u_{yy}=0$.

(14) The distance between two points on a sphere is given by

$$ds=\sqrt{a^2\,d\theta^2+a^2\sin^2\theta\,d\phi^2},$$

Show that the shortest curve between the two points on a sphere lies on a great circle.

(15) The kinetic and potential energies of a spherical pendulum are given by

$$T=\tfrac{1}{2}(l^2\dot\theta^2+l^2\sin^2\theta\,\dot\phi^2), \qquad V=mgl(1-\cos\theta).$$

Find the equation of motion.

(16) Complete the proof of Theorem 8.6.1 by showing that the equation $Ax-\lambda x=y$ has a solution given by (8.6.14) if only α is not an eigenvalue of A and $\alpha\neq 0$.

(17) Show that the Chebyshev polynomial T_n is even for even n, and odd for odd n.

(18) Show that $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$ for $n=2, 3, \ldots$.

(19) Show that

$T_0(x)=1,$

$T_1(x)=x,$

$T_2(x)=2x^2-1,$

$T_3(x)=4x^3-3x,$

$T_4(x)=8x^4-8x^2+1,$

$T_5(x)=16x^5-20x^3+5x.$

(20) Find the best approximation of $f(x)=x^{n+2}$ out of Π_n.

(21) Prove that, for every $n\in\mathbf{N}$, T_n satisfies the differential equation

$$(1-x^2)y''(x)-xy'(x)+n^2y(x)=0.$$

(22) Show that every $P(x)\in\Pi_n$ has a unique representation of the form

$$P(x)=\sum_{k=0}^{n} a_k T_k(x).$$

(23) Show that

$$T_n(x)=x^n-\binom{n}{2}x^{n-2}(1-x^2)+\binom{n}{4}x^{n-4}(1-x^2)^2-\cdots.$$

(24) (a) Let $L_n f$ denote the polynomial of degree less than or equal to n which agrees with a given function $f\in\mathscr{C}([a,b])$ at the fixed nodes $x_0,x_1,\ldots,x_n\in[a,b]$. Show that L_n is a linear operator on $\mathscr{C}([a,b])$.

(b) Show that $L_nP=P$ if and only if P is a polynomial of degree less than or equal to n.

(c) Show that the error in the Lagrange interpolation is

$$(L_nf-f)(x)=\sum_{k=0}^{n}[f(x_k)-f(x)]l_k(x).$$

where

$$l_k(x)=\prod_{\substack{k=0\\k\neq j}}^{n}[(x-x_k)/(x_j-x_k)]$$

(25) Prove that the Lagrange interpolating polynomial for nodes defined as zeros of $T_n(x)$ is

$$P(x)=\frac{1}{n}\sum_{k=0}^{n} f(x_k)\,\frac{T_n'(x)(-1)^{k-1}\sin\theta_k}{x-x_k},$$

where $x_k=\cos\theta_k$ and $\theta_k=(2k-1)\pi/2n$.

(26) Show that the least-squares approximation of degree $n-1$ to $f(x)=x^n$ is $Q_{n-1}^*=x^n-\tilde{P}_n(x)$, where $\tilde{P}_n$ is the orthogonal polynomial of degree n determined by $w(x)$ and (8.7.16).

(27) Show that another form of (8.7.18) is

$$b_k=\frac{(\tilde{P}_k,\tilde{P}_k)}{(\tilde{P}_{k-1},\tilde{P}_{k-1})}.$$

(28) Prove that, if $w(x)=1$, the orthogonal polynomials defined by (8.7.16) satisfy

$$(\tilde{P}_n,\tilde{P}_n)=\frac{2}{2n+1}\tilde{P}_n^2(1),\qquad n=0,1,2,\ldots.$$

(29) Prove Rodrigues' formula:

$$P_n(x)=\frac{1}{2^n n!}D^n[(x^2-1)^n],$$

where P_n is the nth degree Legendre polynomial.

(30) Show that the Legendre polynomials can be represented in the following form:

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}.$$

Use this result to show that

$P_0(x)=1,$

$P_1(x)=x,$

$P_2(x)=\frac{3}{2}x^2-\frac{1}{2},$

$P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x,$

$P_4(x)=\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8}.$

Sketch graphs of these polynomials.

(31) Prove the following recurrence relations for Legendre polynomials:

(a) $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x),$
(b) $P_n'(x)=xP_{n-1}'(x)+nP_{n-1}(x).$

(32) Show that $(P_n,P_n)=2/(2n+1)$, $n=0,1,2,\ldots.$

(33) Show that

(a) $P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x),$
(b) $xP_n'(x)-P_{n-1}'(x)=nP_n(x).$

(34) Show that

(a) $(x^2-1)P_n'(x)=nxP_n(x)-nP_{n-1}(x),$

(b) $\dfrac{1-x^2}{n^2}[P_n'(x)]^2+[P_n(x)]^2=\dfrac{1-x^2}{n^2}[P_{n-1}'(x)]^2+[P_{n-1}(x)]^2.$

(35) Show that

(a) $T_n(x)=U_n(x)-xU_{n-1}(x),$
(b) $(1-x^2)U_{n-1}(x)=xT_n(x)-T_{n+1}(x).$

(36) Show that $U_n(x)$ is generated by the following three-term recurrence relation:

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \qquad n \geq 2,$$
$$U_0(x) = 1, \qquad U_1(x) = 2x.$$

(37) Find the general solution of the system

$$\dot{x} = y + x(1 - x^2 - y^2), \qquad \dot{y} = -x + y(1 - x^2 - y^2).$$

Show that the unit circle is the orbit of a solution of the system.

(38) Show that the equilibrium points of the Volterra system

$$\dot{x} = ax - Ax^2 - bxy, \qquad \dot{y} = -cy + dxy$$

are

$$(0, 0), \left(\frac{a}{A}, 0\right), \left(\frac{c}{d}, \frac{a}{b}\left(1 - \frac{Ab}{ad}\right)\right).$$

(39) Consider a diffusion–reaction system

$$u_t = Lu, \qquad u(0, t) = u(1, t) = 0,$$

in a Hilbert space $L^2([0, 1])$ where $Lu = u_{xx} + a(x)u$. If a is a constant, show that the eigenvalues of L are $a - n^2\pi^2$, $n = 1, 2, 3, \ldots$. Hence show that the equilibrium solution is stable or unstable according as $a < \pi^2$ or $a > \pi^2$. If a is a not a constant, discuss the stability of the system.

(40) Find the non-trivial solutions of the linear eigenvalue problem

$$w'' + \lambda w = 0, \qquad 0 \leq x \leq \pi,$$
$$w(0) = w(\pi) = 0.$$

Draw the bifurcation diagram.

(41) Find small non-trivial solutions of the nonlinear eigenvalue problems

(a) $w'' + \left[\lambda - \frac{2}{\pi}\int_0^{\pi} w^2(x)\, dx\right] w(x) = 0, \qquad 0 \leq x \leq \pi;$

(b) $\left[\frac{2}{\pi}\int_0^{\pi} w^2(x)\, dx\right] w'' + \lambda w = 0, \qquad 0 \leq x \leq \pi;$

with the boundary conditions $w(0) = w(\pi) = 0$. Draw the bifurcation diagrams in each case.

(42) Find small non-trivial solutions of the nonlinear eigenvalue problem

$$u'' + \lambda u - u^2 = 0, \qquad 0 \leq x \leq 1,$$
$$u(0) = u(1) = 0.$$

Discuss their behavior as a function of λ.

(43) The Euler elastic equation for the displacement of a thin elastic rod with end-shortening proportional to λ is

$$w'' + \left(\lambda - \frac{1}{2}\int_0^1 (w')^2\, ds\right) w = 0, \qquad 0 \leq x \leq 1,$$
$$w(0) = w(1) = 0.$$

Describe the behavior of the solution as a function of λ.

(44) Consider the nonlinear boundary value problem for a pinned inextensible rod subject to prescribed axial thrust. The shape of the rod is determined by $\theta(x)$, the angle between the centerline of the deformed rod and the x-axis, and the displacements $u(x)$ and $w(x)$ parallel and normal to the x-axis, respectively. The governing equations of the problem for the elastica are

$$\theta'' + \lambda \sin\theta = 0, \qquad 0 \leq x \leq 1,$$
$$\theta'(0) = \theta'(1) = 0,$$
$$u' = \cos\theta - 1, \qquad w' = \sin\theta, \qquad 0 \leq x \leq 1,$$
$$u(0) = w(0) = w(1) = 0,$$

where the constant λ is proportional to the applied thrust.

Find the eigenvalues and eigenfunctions of the linearized problem. Show that the linearized eigenvalue problem yields the points of bifurcation for the nonlinear problem.

(45) The nonlinear integrodifferential system

$$u'' + \left[\lambda - 2\int_0^1 u^2(x)\, dx\right] u = 0, \qquad u(0) = u(1) = 0,$$

can be solved exactly. Describe its solutions as a function of λ.

(46) Show that the solution of nonlinear integral equation

$$u = 1 + \lambda \int_0^1 u^2\, dx$$

is

$$u = \frac{1}{2\lambda}[1 \pm \sqrt{1-4\lambda}].$$

Draw the graph of u as a function of λ.

(47) Show that the following boundary value problem has no bifurcation from any eigenvalue of the linearized problem:

$$\ddot{u} + \lambda[u + v(u^2+v^2)] = 0, \qquad \ddot{v} + \lambda[v - u(u^2+v^2)] = 0,$$
$$u(0) = u(a) = v(0) = v(a) = 0.$$

(48) Consider the nonlinear system

$$\dot{x} = y - \tfrac{1}{2}x(x^2+y^2), \qquad \dot{y} = -x - \tfrac{1}{2}y(x^2+y^2).$$

Show that the linearized system has the periodic solution $(x, y) = (a \cos t, a \sin t)$, but the nonlinear system has no non-trivial periodic solution.

(49) Discuss the stability and bifurcation phenomena for the following differential equations:

$$\text{(a)}\ \dot{x} = ax \pm x^2, \qquad \text{(b)}\ \dot{x} = ax \pm x^3.$$

(50) Solve the equation

$$\dot{x} = 1 - x^2, \qquad x(t_0) = x_0 \in (-1, 1).$$

Examine the stability of the solution $x(t) = -1$ for all t.

Hints and Answers to Selected Exercises

Hints and Answers to 1.9. Exercises

(5) Consider the function $f(x) = (1/p)x + 1/q - x^{1/p}$. Show that f is decreasing on $[0, 1]$ and $f(1) = 0$.

(6) If $|x| < 1$, then $|x^q| \le |x|^p$. Consider the sequence $\{n^{-1/p}\}$.

(7) The system

$$\lambda_1 = x_1,$$
$$\lambda_1 + \lambda_2 = x_2,$$
$$\lambda_1 + \lambda_2 + \lambda_3 = x_3$$

has a solution for any $(x_1, x_2, x_3) \in \mathbf{R}^3$.

(9) Prove that $\lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n = 0$, for all x, implies $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$.

(10) Prove that $\lambda_0 + \lambda_1 e^x + \lambda_2 e^{2x} + \cdots + \lambda_n e^{nx} = 0$, for all x, implies $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$.