

HW #9

5.8
6.1 1, 7, 17
1, 5

1. The general solution of the associated homogeneous equation, $y'' + 3y' + 2y = 0$, is $y = C_1 e^{-t} + C_2 e^{-2t}$. Thus, we put

$$\psi(t) = w_1(t)e^{-t} + w_2(t)e^{-2t},$$

and the system becomes

$$\begin{aligned} w'_1 e^{-t} + w'_2 e^{-2t} &= 0 \\ -w'_1 e^{-t} - 2w'_2 e^{-2t} &= e^{2t}. \end{aligned}$$

Thus $w'_1 = e^{3t}$, $w'_2 = -e^{4t}$. Integrating these expressions yields $w_1 = \frac{1}{3}e^{3t}$ and $w_2 = -\frac{1}{4}e^{4t}$. Then

$$\begin{aligned} \psi(t) &= \frac{1}{3}e^{3t}e^{-t} - \frac{1}{4}e^{-2t} \\ &= \frac{1}{12}e^{2t} \end{aligned}$$

7. The general solution of the associated homogeneous equation, $y'' + 4y = 0$, is $y = C_1 \cos(2t) + C_2 \sin(2t)$. Thus, we put

$$\psi(t) = w_1(t) \cos(2t) + w_2(t) \sin(2t),$$

and the system becomes

$$\begin{aligned} w'_1(t) \cos(2t) + w'_2(t) \sin(2t) &= 0 \\ -2w'_1(t) \sin(2t) + 2w'_2(t) \cos(2t) &= \sin(2t). \end{aligned}$$

Thus $w'_1 = -\frac{1}{2} \sin^2(2t)$, $w'_2 = \frac{1}{2} \sin(2t) \cos(2t)$. Integrating these expressions yields $w_1 = \frac{1}{16}(\sin(4t) - 4t)$ and $w_2 = -\frac{1}{8} \cos^2(2t)$. Then

$$\begin{aligned} \psi(t) &= \frac{1}{16}(\sin(4t) - 4t) \cos(2t) - \frac{1}{8} \cos^2(2t) \sin(2t) \\ &= -\frac{1}{4}t \cos(2t) \end{aligned}$$

17. The general solution of the associated homogeneous equation, $y'' + 2y' + y = 0$, is $y = e^{-t}(C_1 + C_2 t)$. Thus, we put

$$\psi(t) = e^{-t}(w_1(t) + w_2(t)t),$$

and the system becomes

$$\begin{aligned} e^{-t}(w'_1(t) + w'_2(t)t) &= 0 \\ e^{-t}(-w'_1(t) + w'_2(t)(-t + 1)) &= b(t). \end{aligned}$$

Thus $w'_1 = -tb(t)e^t$, $w'_2 = b(t)e^t$. Then

$$\begin{aligned} \psi(t) &= e^{-t}\left(-\int_0^t [sb(s)e^s]ds + \int_0^t [b(s)e^s]ds \cdot t\right) \\ &= e^{-t} \int_0^t [-sb(s)e^s + tb(s)e^s]ds \end{aligned}$$

The general solution is $y = e^{-t}(\int_0^t [-sb(s)e^s + tb(s)e^s]ds + C_1 + C_2 t)$
Substituting $y(0) = 0$, $y'(0) = 0$, we get $C_1 = C_2 = 0$. Thus

$$y = e^{-t} \int_0^t [-sb(s)e^s + tb(s)e^s]ds$$

1.

$$\begin{aligned} L(t-2) &= \int_0^\infty (t-2)e^{-st} dt \\ &= \lim_{B \rightarrow \infty} \int_0^B (t-2)e^{-st} dt \\ &= \lim_{B \rightarrow \infty} \int_0^B -\frac{1}{s}(t-2) de^{-st} \\ &= \lim_{B \rightarrow \infty} \left(-\frac{1}{s}(t-2)e^{-st}\Big|_0^B + \int_0^B \frac{1}{s}e^{-st} dt\right) \\ &= \lim_{B \rightarrow \infty} \left(-\frac{1}{s}(B-2)e^{-Bs} - \frac{2}{s} - \frac{1}{s^2}e^{-st}\Big|_0^B\right) \\ &= -\frac{2}{s} + \frac{1}{s^2} \end{aligned}$$

2.

5.

$$\begin{aligned} L(f(t)) &= \int_0^1 0 \cdot e^{-st} dt + \int_1^2 e^{-st} dt + \int_2^\infty 2e^{-st} dt \\ &= -\frac{1}{s} e^{-st} \Big|_1^2 + \int_2^\infty e^{-st} dt \\ &= \frac{e^{-s} - e^{-2s}}{s} + 2 \lim_{B \rightarrow \infty} \int_2^B e^{-st} dt \\ &= \frac{e^{-s} - e^{-2s}}{s} + 2 \lim_{B \rightarrow \infty} -\frac{1}{s} e^{-st} \Big|_2^B \\ &= \frac{e^{-s} - e^{-2s}}{s} - \frac{2}{s} \lim_{B \rightarrow \infty} (e^{-sB} - e^{-2s}) \\ &= \frac{e^{-s} - e^{-2s}}{s} + 2 \frac{e^{-2s}}{s} = \frac{e^{-s} + e^{-2s}}{s} \end{aligned}$$