5.1

1.

(a) The equation can be expressed as $\sin(t)y'' + \cos(t)y' + 3y = 0$. It's a linear differential equation.

(b) This is a linear differential equation.

(c) The term $t/y$ is not linear.Thus this is not a linear differential equation.

(d) This is a linear differential equation.

2.

(a) Divide through by the coefficient of $y''$ to obtain $y'' - \frac{2t}{1-t^2}y' + \frac{6}{1-t^2}y = 0$. The singular points are the points where either $p(t) = -\frac{2t}{1-t^2}$ or $q(t) = \frac{6}{1-t^2}$ are discontinuous. Thus, $t = \pm 1$, where $1 - t^2 = 0$, are singular.

(b) The singular points are the points where $q(t) = -t$ are discontinuous. Thus, there is no singular points.

(c) Divide through by the coefficient of $y''$ to obtain $y'' - \frac{t}{\cos(t)}y' + \frac{t^2-1}{\cos(t)}y = \frac{1}{\cos(t)}$. The singular points are the points where either $p(t) = -\frac{t}{\cos(t)}$ or $q(t) = \frac{t^2-1}{\cos(t)}$ are discontinuous. Thus, $t = (k + \frac{1}{2})\pi$, where $k$ is a integer, are singular points.

(d) The singular points are the points where $q(t) = \tan(t)$ are discontinuous. Thus, $t = (k + \frac{1}{2})\pi$, where $k$ is a integer, are singular points.

7. Let $\psi(t) = \phi_2'(0)\phi_1(t) - \phi_1'(0)\phi_2(t)$. Then $\psi(t)$ is a solution of the ODE, and since $\phi_1(0) = \phi_2(0) = 0$, $\psi(0) = 0$. Since $\psi'(t) = \phi_2'(0)\phi_1'(t) - \phi_1'(0)\phi_2'(t)$, it follows that $\psi'(0) = 0$. By corollary 5.1.1 $\psi(t) \equiv 0$

5.2

1. Let $p(t)$ be a function that has a fourth derivative on $\mathbf{R}$ (polynomials satisfy this requirement). Then by Taylor's theorem, for any real $t$ there is a number $\xi$ between 1 and $t$ such that

10. Since $\mathcal{L}$ is linear,

$$\begin{aligned}
\mathcal{L}(C_1y_1(t) + C_2y_2(t)) &= C_1\mathcal{L}(y_1(t)) + C_2\mathcal{L}(y_2(t)) \\
&= C_1f(t) + C_2f(t) \\
&= (C_1 + C_2)f(t).
\end{aligned}$$

Thus $C_1y_1(t) + C_2y_2(t)$ is a solution if and only if $C_1 + C_2 = 1$.

14. If $y = e^{-3t}$, then $y' = -3e^{-3t}$ and $y'' = 9e^{-3t}$. Thus

$$y'' + 4y' + 3y = 9e^{-3t} - 12e^{-3t} + 3e^{-3t} = 0.$$

If $y = e^{-t}$, then $y' = -e^{-t}$ and $y'' = e^{-t}$. Thus

$$y'' + 4y' + 3y = e^{-t} - 4e^{-t} + 3e^{-t} = 0.$$

On the other hand, $W[e^{-3t}, e^{-t}] = \det \begin{bmatrix} e^{-3t} & e^{-t} \\ -3e^{-3t} & -e^{-t} \end{bmatrix} = 2e^{-4t} \neq 0.$
Therefore $S$ is a fundamental set of solutions. The general solution is

$$y = c_1 e^{-3t} + c_2 e^{-t}$$

5.3

3. The characteristic equation is $s^2 = 0$. There is a double characteristic root, 0. The general solution is $y = C_1 t + C_2$.

5. The characteristic equation is $s^2 + 25 = 0$. The characteristic roots are $\pm 5i$. The general solution is $y = C_1 \cos(5t) + C_2 \sin(5t)$.

13. The characteristic equation is $2s^2 + s - 1 = 0$. The characteristic roots are $\frac{1}{2}, -1$. The general solution is $y = C_1 e^{t/2} + C_2 e^{-t}$.

17. Let $A = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$. Then $\operatorname{tr}(A) = -p$, $\det(A) = q$. Thus the characteristic equation of the equivalent system is $s^2 + ps + q$. It's same as the characteristic equation of the ODE.

5.5

3. Put $y = Ae^{2t}$, then $y' = 2Ae^{2t}$, and $y'' = 4Ae^{2t}$. Substituting these in the ODE, we find

$$4Ae^{2t} + 6Ae^{2t} + 2Ae^{2t} = e^{2t}$$

Cancelling $e^{2t}$ and simplifying shows $12A = 1$, so $A = \frac{1}{12}$. The particular solution is

$$y_p = \frac{1}{12} e^{2t}$$

4. Let $\mathcal{L}(y) = y'' + 3y' + 2y$. The characteristic roots of $\mathcal{L}(y) = 0$ are $-1$ and $-2$. Let us start by finding a particular solution $y_1(t)$ of $\mathcal{L}(y) = e^{2t}$. Put $y_1 = Ae^{2t}$. Then

$$\mathcal{L}(y_1) = A(4e^{2t} + 6e^{2t} + 2e^{2t}) = A(12e^{2t}) = e^{2t}$$

so $A = \frac{1}{12}$ and $y_1 = \frac{1}{12}e^{2t}$.
A particular solution $y_2(t)$ of $\mathcal{L}(y) = -e^{-t}$ will have the form $y_2 = Bte^{-t}$, where

$$\mathcal{L}(y_2) = B((te^{-t} - 2e^{-t}) + 3(-te^{-t} + e^{-t}) + 2te^{-t}) = B(e^{-t}) = -e^{-t}.$$

Thus $B = -1$ and $y_2 = -te^{-t}$. Therefore

$$y_p = y_1 + y_2 = \frac{1}{12}e^{2t} - te^{-t}$$

11. Let $\mathcal{L}(y) = y'' - 4y' + 2y$. The characteristic roots of $\mathcal{L}(y) = 0$ are $2 \pm \sqrt{2}$. Thus the general solution of $\mathcal{L}(y) = 0$ is $y_h = C_1e^{(2+\sqrt{2})t} + C_2e^{(2-\sqrt{2})t}$. Let $y_p(t) = Ae^{2t}$ be a particular solution. Then $y_p' = 2Ae^{2t}$, $y_p'' = 4Ae^{2t}$, and therefore $\mathcal{L}(y_p) = -2Ae^{2t} = e^{2t}$. Thus $A = -\frac{1}{2}$. It follows that $y_p(t) = -\frac{1}{2}e^{2t}$, and the general solution is

$$y = -\frac{1}{2}e^{2t} + C_1e^{(2+\sqrt{2})t} + C_2e^{(2-\sqrt{2})t}.$$

15. Let $\mathcal{L}(y) = y'' - 4y' + 2y$. The characteristic roots of $\mathcal{L}(y) = 0$ are $2 \pm \sqrt{2}$. Thus the general solution of $\mathcal{L}(y) = 0$ is $y_h = C_1e^{(2+\sqrt{2})t} + C_2e^{(2-\sqrt{2})t}$. Let $y_p(t) = Ae^{2t}$ be a particular solution. Then $y_p' = 2Ae^{2t}$, $y_p'' = 4Ae^{2t}$, and therefore $\mathcal{L}(y_p) = -2Ae^{2t} = e^{2t}$. Thus $A = -\frac{1}{2}$. It follows that $y_p(t) = -\frac{1}{2}e^{2t}$, and the general solution is $y = -\frac{1}{2}e^{2t} + C_1e^{(2+\sqrt{2})t} + C_2e^{(2-\sqrt{2})t}$, and $y' = -e^{2t} + (2+\sqrt{2})C_1e^{(2+\sqrt{2})t} + (2-\sqrt{2})C_2e^{(2-\sqrt{2})t}$. Substituting $y(0) = 0$, and $y'(0) = 0$, we have
$\begin{cases} -\frac{1}{2} + C_1 + C_2 = 0 \\ -1 + (2+\sqrt{2})C_1 + (2-\sqrt{2})C_2 = 0 \end{cases}$ Thus $C_1 = \frac{1}{4}$, and $C_2 = \frac{1}{4}$ and the solution is

$$y = -\frac{1}{2}e^{2t} + \frac{1}{4}e^{(2+\sqrt{2})t} + \frac{1}{4}e^{(2-\sqrt{2})t}$$

21. Let $\mathcal{L}(y) = y'' + 2y' + 2y$. The characteristic roots of $\mathcal{L}(y) = 0$ are $-1 \pm i$. Thus the general solution of $\mathcal{L}(y) = 0$ is $y_h = e^{-t}(C_1\cos(t) + C_2\sin(t))$.

Let $y_p(t) = A\cos(t) + B\sin(t)$ be a particular solution. Then $y_p' = -A\sin(t) + B\cos(t)$, $y_p'' = -A\cos(t) - B\sin(t)$ and therefore $\mathcal{L}(y_p) = -A\cos(t) - B\sin(t) - 2A\sin(t) + 2B\cos(t) + 2A\cos(t) + 2B\sin(t) = (A+2B)\cos(t) + (B-2A)\sin(t) = 10\sin(t)$. We get $A + 2B = 0$, and $B - 2A = 10$. Thus $A = -4$, and $B = 2$ It follows that $y_p(t) = -4\cos(t) + 2\sin(t)$, and the general solution is $y = -4\cos(t) + 2\sin(t) + e^{-t}(C_1\cos(t) + C_2\sin(t))$, Hence $y' = 4\sin(t) + 2\cos(t) + e^{-t}(-C_1\cos(t) - C_2\sin(t) - C_1\sin(t) + C_2\cos(t))$. Substituting $y(0) = 0$, and $y'(0) = -6$, we obtain

$\begin{cases} -4 + C_1 = 0 \\ 2 - C_1 + C_2 = -6 \end{cases}$ Thus $C_1 = 4$, and $C_2 = -4$. The solution is

$$y = -4\cos(t) + 2\sin(t) + e^{-t}(4\cos(t) - 4\sin(t))$$