4.4.4. *Find a fundamental matrix solution of the system*

$$\begin{cases} x' = -x - y \\ y' = x - 3y \end{cases}$$

Let $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the above becomes $\mathbf{v} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$, and the two independent solutions for $\mathbf{v}$ will be the columns of the fundamental matrix χ. We determine $\mathbf{v}$ by taking the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$. The characteristic polynomial is $s^2 + 4s + 4$, so there is a repeated eigenvalue of $s = -2$. Solving $A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ gives the pair of equations $-c_1 - c_2 = -2c_1$ and $c_1 - 3c_2 = -2c_1$, both of which simplify to the relationship $c_1 = c_2$, so arbitrarily letting $c_1 = 1$ we get the sole eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution in such a case also requires a non-eigenvector, so we find one arbitrarily letting $c_1 \neq c_2$, for instance $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then the two independent solutions are, following the formula for solutions of matrices with double roots,

$$\mathbf{v}_1 = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$

$$\mathbf{v}_2 = e^{-2t} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} e^{-2t} t \\ e^{-2t}(t+1) \end{bmatrix}$$

Using these as the columns of the matrix solution, we get

$$\chi = e^{-2t} \begin{bmatrix} 1 & t \\ 1 & t+1 \end{bmatrix}$$

Note that other solutions are possible, using different arbitrary non-eigenvalues.

4.4.8. *Use the method of variation of constants to find a particular solution of the system of ODEs*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix},$$

and then write down the general solution. The fundamental matrix solution of the associated homogeneous system was found in Exercise 4.4.4.

We will be finding a solution by letting $\mathbf{v}_p(t) = \chi(t)\mathbf{w}(t)$. Substituting this in for $\begin{bmatrix} x \\ y \end{bmatrix}$, and using A as given in Exercise 4.4.4, we will find that $\mathbf{v}_p' - A\mathbf{v}_p = \mathbf{b}$. Expanding $\mathbf{v}_p = \chi\mathbf{w}$ and $\mathbf{v}_p' = \chi'\mathbf{w} + \chi\mathbf{w}'$, we find that

$$\chi\mathbf{w}' + \chi'\mathbf{w} - A\chi\mathbf{w} = \mathbf{b}$$

We constructed χ as a matrix solution to $\chi' = A\chi$, so $\chi'\mathbf{w} - A\chi\mathbf{w} = 0$. Thus, the above can be simplified to

$$\chi\mathbf{w}' = \mathbf{b}$$

or in other words,

$$e^{-2t}\begin{bmatrix} w_1' + w_2't \\ w_1' + w_2'(t+1) \end{bmatrix} = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

(a) *Find the general solution where* $\begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} = e^{-2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

In this case the above equation, after cancelling e^{-2t}, simplifies to the system

$$\begin{aligned} w_1' + w_2't &= 1 \\ w_1' + w_2'(t+1) &= 3 \end{aligned}$$

Subtracting these equations from each otheryields that $w_2' = 2$, and substituting that back into either of the original equations gives $w_1' = 1 - 2t$; on integration it follows that $w_1 = t - t^2 + C$ and $w_2 = 2t + D$, so the general solution of the differential equation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{v}_p = \chi\begin{bmatrix} t - t^2 + C \\ 2t + D \end{bmatrix} = e^{-2t}\begin{bmatrix} t^2 + t + C + Dt \\ t^2 + 3t + C + Dt + D \end{bmatrix}$$

(b) *Find the general solution where* $\begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-2t} \\ 0 \end{bmatrix}$.

In this case the above equation, after cancelling e^{-2t}, simplifies to the system

$$\begin{aligned} w_1' + w_2't &= 2 \\ w_1' + w_2'(t+1) &= 0 \end{aligned}$$

Subtracting these equations from each otheryields that $w_2' = -2$, and substituting that back into either of the original equations gives $w_1' = 2 + 2t$; on integration it follows that $w_1 = t^2 + 2t + C$ and $w_2 = -2t + D$, so the general solution of the differential equation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{v}_p = \chi\begin{bmatrix} t^2 + 2t + C \\ -2t + D \end{bmatrix} = e^{-2t}\begin{bmatrix} -t^2 + 2t + C + Dt \\ t^2 + C + Dt + D \end{bmatrix}$$

4.4.10. *Find the general solution of each of the following inhomogeneous systems.*

(a)

$$\begin{cases} x' = x + y + t^{-1} \\ y' = -x - y \end{cases}$$

(c)

The homogeneous associated problem, in matrix form, is

$$\mathbf{v}_h' = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{v}_h$$

Let us call the matrix above A. The characteristic polynomial of A is s^2, so A has a repeated eigenvalue of 0. Solving the equation $A\mathbf{b} = 0$ requires that $b_1 + b_2 = 0$, so arbitrarily we can find the eigenvector $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and choose the non-eigenvector $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, again arbitrarily. Then the two linearly independent solutions to the homogeneous problem are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the matrix with these solutions as columns is

$$\chi = \begin{bmatrix} 1 & t \\ -1 & 1-t \end{bmatrix}$$

So now, let $\mathbf{v}_p = \chi\mathbf{w}$, and by the same logic as in Exercise 4.4.8, we know that $\chi\mathbf{w}' = \mathbf{r}$, where $\mathbf{r}$ is the vector of nonhomogeneous terms, so in this case:

$$\begin{bmatrix} w_1' + w_2' t \\ -w_1' + (1-t)w_2' \end{bmatrix} = \begin{bmatrix} t^{-1} \\ 0 \end{bmatrix}$$

So by the second row, $w_1' = (1-t)w_2'$, which plugged into the first row yields that $w_2' = t^{-1}$; thus $w_1' = \frac{1-t}{t} = t^{-1} - 1$. Integrating each of these, we find that $w_1 = \ln|t| - t + C$ and $w_2 = \ln|t| + D$. Thus the solution is

$$\mathbf{v} = \chi\mathbf{w} = \chi = \begin{bmatrix} 1 & t \\ -1 & 1-t \end{bmatrix} \begin{bmatrix} \ln|t| - t + C \\ \ln|t| + D \end{bmatrix} = \begin{bmatrix} (1+t)\ln|t| - t + C + Dt \\ -t\ln|t| + t - C + D - Dt \end{bmatrix}$$

4.4.11. *Let C be a constant matrix, and suppose that $\chi(t)$ is a matrix solution of $\mathbf{v}' = A(t)\mathbf{v}$. Show that χC is also a matrix solution.*

$\chi(t)$ is a matrix solution of $\mathbf{v}' = A(t)\mathbf{v}$ if and only if $\chi'(t) = A(t)\chi(t)$, from which it follows that $\chi'(t)C = A(t)\chi(t)C$, which we can rewrite as $\frac{d}{dt}(\chi(t)C) = A(t)(\chi(t)C)$, so $\chi(t)C$ is also a matrix solution.

2.2.1. *Decide if the ODE $(2x + 5y + 3)dx + (5x - 4y + 2)dy = 0$ is exact; if it is, find an integral.*

$\frac{\partial}{\partial y}(2x+5y+3) = 5$ and $\frac{\partial}{\partial x}(5x-4y+2) = 5$, so since these are equal, the ODE is exact. The integral will have the form $F(x,y) = \int 2x + 5y + 3dx = x^2 + 5xy + 3x + H(y)$. We know that $\frac{\partial F}{\partial y}$ must be $5x - 4y + 2$, so since $\frac{\partial F}{\partial y} = 5x + H'(y)$, it follows that $H'(y) = 2 - 4y$, so $H(y) = 2y - 2y^2 + C$. Thus since $F(x,y)$ is a constant, the solution is $x^2 + 5xy + 3x + 2y - 2y^2 + C = 0$.

2.2.2. *Decide if the ODE $ydx + (x+y)dy = 0$ is exact; if it is, find an integral.*

$\frac{\partial}{\partial y}y = 1$ and $\frac{\partial}{\partial x}(x+y) = 1$. Since these are equal, the ODE is exact. The integral will have the form $F(x,y) = \int y dx = xy + H(y)$. We know that $\frac{\partial F}{\partial y}$ must be $x+y$, so since $\frac{\partial F}{\partial y} = x + H'(y)$, it follows that $H'(y) = y$, so $H(y) = \frac{1}{2}y^2 + C$. Thus since $F(x,y)$ is a constant, the solution is $xy + \frac{1}{2}y^2 + C = 0$.

2.2.15. *Find an integrating factor for the ODE $(x^2 + xy^2 + 1)dx + 2ydy = 0$, and use it to determine an integral.*

Let us multiply both sides by some $m(x)$, then compare the partials. $\frac{\partial}{\partial y}[m(x)(x^2 + xy^2 + 1)] = 2m(x)xy$, while $\frac{\partial}{\partial x}(2m(x)y) = 2ym'(x)$. Thus, for exactnes,s it must be true that $2m(x)xy = 2ym'(x)$, which, with terms cancelled, becomes $m'(x) = xm(x)$, so using the solution to a linear (or separable) differential equation we can find that $m(x) = e^{\frac{x^2}{2}}$. Then our differential equation, in exact form, becomes

$$(x^2 + xy^2 + 1)e^{\frac{x^2}{2}}dx + 2ye^{\frac{x^2}{2}}dy = 0$$

We will perform the easier of the two integrations, and let $F(x,y) = \int 2ye^{\frac{x^2}{2}}dy = y^2e^{\frac{x^2}{2}} + H(x)$ We know that $\frac{\partial F}{\partial x}$ must be $(x^2+xy^2+1)e^{\frac{x^2}{2}}$, so since $\frac{\partial F}{\partial x} = xy^2e^{\frac{x^2}{2}} + H'(x)$, it follows that $H'(x) = (x^2+1)e^{\frac{x^2}{2}}$. Integrating this is rather involved, we start by integrating $\int x^2 e^{\frac{x^2}{2}}dx$ by parts with $u = x$ and $dv = xe^{\frac{x^2}{2}}dx$ to get $xe^{\frac{x^2}{2}} - \int e^{\frac{x^2}{2}}dx$; rearranging, we find that $\int x^2e^{\frac{x^2}{2}} + e^{\frac{x^2}{2}}dx = xe^{\frac{x^2}{2}} + C$, so that is the value of $H(x)$. Thus the solution of the integral is $x(y^2+1)e^{\frac{x^2}{2}} + C = 0$.

2.2.16. *Find an integrating factor for the ODE $xdy - (y-x)dx = 0$, and use it to determine an integral.*

Multiplying each term by $m(x)$ and taking partials, we find that $\frac{\partial}{\partial x}(m(x)x) = m(x) + xm'(x)$ and $\frac{\partial}{\partial y}[-(y-x)m(x)] = -m(x)$, so our exactness criterion is that $m(x) + xm'(x) = -m(x)$, so $m'(x) = -\frac{2}{x}m(x)$, which is a linear (or separable) differential equation with solution $m = x^{-2}$, so the exact form of the ODE is $\frac{1}{x}dy - (\frac{y}{x^2} - \frac{1}{x})dx = 0$. Thus $F(x,y) = \int \frac{1}{x}dy = \ln|x| + H(x)$. We know that $\frac{\partial F}{\partial x}$ must be $\frac{y}{x^2} - \frac{1}{x}$.

2.2.22. *Show that if $y' = f(x,y)$ is a separable ODE, with $f(x,y) = g(x)h(y)$, then $[h(y)]^{-1}$ is an integrating factor for*

$$dy - f(x,y)dx = 0.$$

Thus, the method for solving separable ODEs is a special case of the integrating factor method.

Multiplying $dy - g(x)h(y)dx = 0$ by the factor $\frac{1}{h(y)}$ yields $\frac{dy}{h(y)} - g(x)dx = 0$. To show that this is exact, note that $\frac{\partial}{\partial x}\frac{1}{h(y)} = 0$ and $\frac{\partial}{\partial y}(-g(x) = 0$. Since these are equal, the resulting differential equation is exact, so our multiplier served as an integrating factor.

2.2.23. *Write the linear ODE, $y' + p(x)y = q(x)$ in the equivalent form*

$$dy + [p(x)y - q(x)]dx = 0,$$

and find a one-variable integrating factor $m(x)$*.*

Considering $m(x)dy + [p(x)y - q(x)]m(x)dx = 0$, let us observe that $\frac{\partial}{\partial x}m(x) = m'(x)$ and

$$\frac{\partial}{\partial y}[p(x)y - q(x)]m(x) = p(x)m(x)$$

. Thus our $m(x)$ is subject to the differential equation $m'(x) = p(x)m(x)$, which is a linear (or separable) differential equation with solution $m(x) = e^{\int p(x)dx}$.

4.3.1. *Find the reciprocal of* $2 + i$*.*

We multiply by the complex conjugate to make the denominator real:

$$\frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i$$

4.3.3. *Find the sixth roots of* 1 *and locate them on the complex plane.*

$1 = e^{2\pi ni}$, so $\sqrt[6]{1} = e^{\frac{2\pi ni}{6}} = e^{\frac{\pi}{3}n}$. For values of n from 0 to 5, this takes on the following values: 1, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, -1, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

4.3.5. *Show that* $|e^{\lambda+i\omega}| = e^{\lambda}$

$|e^{\lambda+i\omega}| = |e^{\lambda}(\cos\omega + i\sin\omega)| = |e^{\lambda}| \cdot |\cos\omega + i\sin\omega|$. Notice that $|\cos\omega + i\sin\omega| = \sqrt{\cos^2\omega + \sin^2\omega} = 1$, so the above is simply $|e^{\lambda}|$, and since λ is real, this is positive, so it is simply e^{λ}.