Homework # 2 Solutions

11.4

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)

   (b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

   (b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

16. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\left(p = \frac{3}{2} > 1\right)$.

[illegible]. $\dfrac{n!}{n^n} = \dfrac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n}{n \cdot n \cdot n \cdot \cdots \cdot n \cdot n} \le \dfrac{1}{n} \cdot \dfrac{2}{n} \cdot 1 \cdot 1 \cdot \cdots \cdot 1$ for $n \ge 2$, so since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges $(p = 2 > 1)$, $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges also by the Comparison Test.

37. Since $\dfrac{d_n}{10^n} \le \dfrac{9}{10^n}$ for each n, and since $\displaystyle\sum_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series $\left(|r| = \frac{1}{10} < 1\right)$, $0.d_1 d_2 d_3 \ldots = \displaystyle\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

[illegible]. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n \to \infty$, so for some integer N, $a_n \le 1$ for all $n \ge N$. But then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} a_n b_n \le \sum_{n=1}^{N-1} a_n b_n + \sum_{n=N}^{\infty} b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty} b_n$ converges. So $\sum a_n b_n$ converges by the Comparison Test.

11.5

8. $b_n = \dfrac{2n}{4n^2+1} > 0$, $\{b_n\}$ is decreasing [since

$$b_n - b_{n+1} = \frac{2n}{4n^2+1} - \frac{2n+2}{4n^2+8n+5} = \frac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)} > 0 \text{ for } n \ge 1], \text{ and}$$

$\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2/n}{4+1/n^2} = 0$, so the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2+1}$ converges by the Alternating Series Test.

Alternatively, to show that $\{b_n\}$ is decreasing, we could verify that $\dfrac{d}{dx}\left(\dfrac{2x}{4x^2+1}\right) < 0$ for $x \ge 1$.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \dfrac{1}{n^{3/4}}$ is decreasing and positive and $\displaystyle\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

24. The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)^4} < \dfrac{1}{n^4}$ and

    (ii) $\displaystyle\lim_{n\to\infty} \frac{1}{n^4} = 0$, so the series is convergent. Now $b_5 = 1/5^4 = 0.0016 > 0.001$ and

    $b_6 = 1/6^4 \approx 0.00077 < 0.001$, so by the Alternating Series Estimation Theorem, $n = 5$.

11.6

7. $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{5+n} = \lim_{n\to\infty} \frac{1}{5/n+1} = 1$, so $\lim_{n\to\infty} a_n \neq 0$. Thus, the given series is divergent by the Test for Divergence.

16. $n^{2/3} - 2 > 0$ for $n \geq 3$, so $\frac{3-\cos n}{n^{2/3}-2} > \frac{1}{n^{2/3}-2} > \frac{1}{n^{2/3}}$ for $n \geq 3$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges $\left(p = \frac{2}{3} \leq 1\right)$, so does $\sum_{n=1}^{\infty} \frac{3-\cos n}{n^{2/3}-2}$ by the Comparison Test.

33. (a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ by Theorem 12.2.6 [ET 11.2.6].

37. Summing the inequalities $-|a_i| \leq a_i \leq |a_i|$ for $i = 1, 2, \ldots, n$, we get $-\sum_{i=1}^{n} |a_i| \leq \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} |a_i|$ $\Rightarrow$ $-\lim_{n\to\infty} \sum_{i=1}^{n} |a_i| \leq \lim_{n\to\infty} \sum_{i=1}^{n} a_i \leq \lim_{n\to\infty} \sum_{i=1}^{n} |a_i|$ $\Rightarrow$ $-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|$ $\Rightarrow$ $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

11.8

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

12. $a_n = \frac{x^n}{5^n n^5}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{5^{n+1}(n+1)^5} \cdot \frac{5^n n^5}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{5}\left(\frac{n}{n+1}\right)^5 = \frac{|x|}{5}$. By the Ratio Test, the series converges when $|x|/5 < 1$ $\Leftrightarrow$ $|x| < 5$, so $R = 5$. When $x = -5$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x = 5$, we get the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ $(p = 5 > 1)$.

Thus, $I = [-5, 5]$.

29. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \le 4$. In particular, it converges when $x = -2$; that is, $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n(-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n/(n4^n)$.]

36. $s_{4n-1} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_0x^4 + c_1x^5 + c_2x^6 + c_3x^7 + \cdots + c_3x^{4n-1}$

$$= \left(c_0 + c_1x + c_2x^2 + c_3x^3\right)\left(1 + x^4 + x^8 + \cdots + x^{4n-4}\right) \to \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4} \text{ as } n \to \infty$$

[by (12.2.4) [ET (11.2.4)] with $r = x^4$] for $|x^4| < 1 \iff |x| < 1$. Also s_{4n}, s_{4n+1}, s_{4n+2} have the same limits (for example, $s_{4n} = s_{4n-1} + c_0x^{4n}$ and $x^{4n} \to 0$ for $|x| < 1$). So if at least one of c_0, c_1, c_2, and c_3 is nonzero, then the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$.

11.9

7. $f(x) = \dfrac{1}{x-5} = -\dfrac{1}{5}\left(\dfrac{1}{1 - x/5}\right) = -\dfrac{1}{5}\sum_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n$ or equivalently, $-\sum_{n=0}^{\infty} \dfrac{1}{5^{n+1}} x^n$. The series converges when $\left|\dfrac{x}{5}\right| < 1$; that is, when $|x| < 5$, so $I = (-5, 5)$.

16. We know that $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n$. Differentiating, we get $\dfrac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n$, so

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2}\sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} 2^n(n+1)x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2}(n-1)x^n,$$

with $R = \frac{1}{2}$.

26. By Example 7, $\displaystyle\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R = 1$.

35. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 10.4.2 [ET 9.4.2], the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.

Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.