HW5 Solution of Math170A

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6.1.1

Proof.

(a).

For any $v_1, v_2 \in S_\lambda$, we have $A(v_1 + v_2) = Av_1 + Av_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2)$, thus $v_1 + v_2 \in S_\lambda$.

(b).

Since for any $v \in S_\lambda$, $A(\alpha v) = \alpha A v = \lambda(\alpha v)$, thus $\alpha v \in S_\lambda$. □

6.2.2

Proof.

(a).

Similar with the proof of 6.1.1.

(b).

Frist, show $A^j S \subset A(A^{j-1}S)$. Pick a $y \in A^j S$, it implies $\exists x \in S$ such that $y = A^j x$. Then, $y = A(A^{j-1})x \in A(A^{j-1}S)$.

Second, show $A^j S \supset A(A^{j-1}S)$. Pick a $y \in A(A^{j-1}S)$, one has $y = A(A^{j-1}x) = A^j x \in A^j S$.

(c).

Frist, show $AS \subset span\{Ax_1, \cdots, Ax_k\}$. Pick a $y \in AS$, then there is a $x = \sum c_j x_j \in S$ such that $y = Ax = A\sum c_j x_j = \sum c_j A x_j \in span\{Ax_1, \cdots, Ax_k\}$.

Second, show $AS \supset span\{Ax_1, \cdots, Ax_k\}$. Pick a $y \in span\{Ax_1, \cdots, Ax_k\}$, then there is a $y = \sum c_j A x_j = A \sum c_j x_j \in AS$.

(d).

Show $\{Ax_1, \cdots, Ax_k\}$ is linear independent set. Consider $\sum c_j A x_j = 0$, take A out of the sum, one has

$A\sum c_j x_j = 0$, since $S \cap N(A) = \{0\}$, we have $\sum c_j x_j \in S \cap N(A)$ which is zero, and because $\{x_1, \cdots, x_k\}$ is linear independent set, thus the coefficients c_js are zeros, hence $\{Ax_1 \cdots, Ax_k\}$ is a linear independent set, by the result of (c), it is a basis of AS.

□

8.1.9

Order the unknowns as shown on the top of page 550 $u^T = [u_{1,1}, \cdots, u_{m-1,1}, u_{1,2}, \cdots, u_{m-1,2}, \cdots]$. Then the matrix A is given in equation (8.1.10) which is tri-diag block matrix.

8.1.12

For the 3D problem, the size of matrix is $(m-1)^3 \times (m-1)^3$, with $(m-1)^3$ unknowns, and the system of equations are

$$\begin{aligned} &For\ \ i,j,k = 1, \cdots, m-1, \\ &6u_{i,j,k} - u_{i-1,j,k} - u_{i+1,j,k} - u_{i,j-1,k} - u_{i,j+1,k} - u_{i,j,k-1} - u_{i,j,k+1} \ = \ h^2 f_{i,j}. \end{aligned} \tag{1}$$

8.2.12

Proof.

(a).

This is a direct result as looking at equation (8.2.9) in the vector form.

(b).

From result of (a),

$$\begin{aligned} x^{k+1} &= D^{-1}(b + Ex^{k+1} + Fx^k) \\ Dx^{k+1} &= b + Ex^{k+1} + Fx^k \\ (D-E)x^{k+1} &= b + Fx^k \\ x^{k+1} &= (D-E)^{-1}(b + Fx^k). \end{aligned} \tag{2}$$

□

(c).

Replace M by $D - E$ and r^k by $b - Ax^k$, we have the formula in (c) is the same as the one in (b).

8.2.24

Do the same work as 8.2.12.

8.3.12

The definition of R_∞ is on page 572. Then,

$$R_\infty(G_{GS}) \;=\; -\log_e \rho(G_{GS}) \;=\; -\log_e \rho(G_J)^2 \;=\; -2\log_e \rho(G_J) \;=\; 2R_\infty(G_J). \tag{3}$$

8.3.14

(a).

Look at the formula in 8.2.12(b), since the flops of matrix-vector multiplication is $O(m^2)$, and vector-vector addition is $O(m)$, thus it is $O(m^2)$.

(b).

After j iterations, the error is decreased by a fixed factor ε means $\|e^{k+j}\|/\|e^k\| \approx \rho(G)^j \le \varepsilon$. For GS method, $R_\infty(G) \approx \pi^2 h^2$ from tabel 8.5 on page573. Since $\varepsilon < 1$ and $h = 1/m$, we have

$$\begin{aligned} \rho(G)^j &\le \varepsilon \\ -j\log_e \rho(G) &\le -\log_e \varepsilon \\ j(\pi^2/m^2) &\le -\log_e \varepsilon \\ j &\le (-\log_e \varepsilon/\pi^2)m^2. \end{aligned} \tag{4}$$

(c).

The overall flops of GS method is the product of (a) and (b), that is $O(m^4)$, which is at the same level of banded Gaussian elimination.

(d).

Do the exactly same thing in (b), and the result is $O(m)$.

(e).

Since from problem 8.2.24(b), the flops of one iteration of SOR is $O(m^2)$, thus overall is $O(m^3)$, thus much less than banded Gaussian elimination.

(f).

Obviously the later is better than SOR.

8.4.12

From 8.4.10, we have

$$\begin{aligned} Ax^{k+1} &= Ax^k + \alpha_k Ap^k \\ b - Ax^{k+1} &= b - Ax^k - \alpha_k Ap^k \\ r^{k+1} &= r^k - \alpha_k Ap^k. \end{aligned} \tag{5}$$

8.4.21

The Richardson's method is introduced on page 569, that is

$$x^{k+1} = (I - \omega A)x^k + \omega b. \tag{6}$$

For steepest descent method, since $p^k = b - Ax^k$, we have

$$x^{k+1} = (I - \alpha_k A)x^k + \alpha_k b. \tag{7}$$

Look at these two methods, the steepest descent method is a Richardson's method with variable damping α_k. And the Richardson's method is steepest descent method with an inexact line search ($\alpha_k = \omega$).

8.7.4

The Preconditioned CG methos is to slove $R^{-T}AR^{-1}(Rx) = R^{-T}b$, where $R^T R = M$. then compare the pseudo code (8.7.3) with (8.7.1).