MATH 171B: Numerical Optimization: Nonlinear Problems

Instructor: Michael Holst

Spring Quarter 2015

Solutions for Homework Assignment #4

Exercise 4.1. *Let $f(x)$ denote a convex continuously differentiable function. Show that if a stationary point x^* exists, then $f(x^*)$ is a global minimum of f. Also show that if $f(x)$ is actually strictly convex, then $x*$ is the unique global minimum. Why can uniqueness be lost if the function is not strictly convex? Draw a picture of such a situation when $f : \mathbb{R} \mapsto \mathbb{R}$.*

If x^* is a stationary point, then, letting $x = x^*$, $f(y) \geq f(x^*) + f'(x^*)(y - x^*) = f(x^*)$ for all y, so $f(x^*)$ is a global minimum of f.

If $f(x)$ is strictly convex, $f(y) > f(x^*) + f'(x^*)(y - x^*) = f(x^*)$ for all y. Assume for contradiction that there exists another global minimizer $\hat{x} \neq x^*$. But then $f(\hat{x}) > f(x^*)$, so $\hat{x}$ cannot be a global minimizer. So x^* is the unique global minimizer.

Exercise 4.2. *This problem requires modifying the Hessian to produce a descent direction. Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.$$

(a) *Derive the gradient $g(x)$ and Hessian $H(x)$ of $f(x)$.*

$$g(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \cos x_3 - e^{x_2} x_3^2 \\ -x_2^2 \sin x_3 - 2e^{x_2} x_3 + 4 \end{pmatrix}, \qquad H(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2\cos x_3 - e^{x_2} x_3^2 & -2x_2 \sin x_3 - 2e^{x_2} x_3 \\ 0 & -2x_2 \sin x_3 - 2e^{x_2} x_3 & -x_2 \cos x_3 - 2e^{x_2} \end{pmatrix}$$

(b) *Compute the spectral decomposition of $H(x)$ at $\bar{x} = (0, 1, 0)^T$.*

Since $H(\bar{x}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -(1+2e) \end{pmatrix}$ is a diagonal matrix, simply let $V = I$. Then $H(\bar{x}) = I \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -(1+2e) \end{pmatrix} I^T$.

(c) *Compute the "pure" Newton direction p^N at $\bar{x}$. Is p^N a descent direction?*

Solving $H(\bar{x})p^N = -g(\bar{x}) = -\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ gives $p^N = \begin{pmatrix} 0 \\ -1 \\ \frac{4}{1+26} \end{pmatrix}$. Since $g^T p^N = -2 + \frac{16}{1+2e} \approx 0.4858 > 0$, p^N is not a descent direction.

(d) *Compute the modified Newton direction p^M at the same point using the eigenvalue reflection technique (the better of the two approaches we discussed in class). Find the directional derivative along p^M at $\bar{x}$.*

$B(\bar{x}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1+2e \end{pmatrix}$. Solving $B(\bar{x})p^M = -g(\bar{x}) = -\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ gives $p^M = \begin{pmatrix} 0 \\ -1 \\ -\frac{4}{1+26} \end{pmatrix}$.

The directional derivative is $g^T \frac{p^M}{||p^M||_2} = \frac{-2-16/1+2e}{\sqrt{1+16/(1+2e)^2}} \approx -3.810.$

(e) *Find a direction of negative curvature at $\bar{x}$. Verify your result numerically.*

We want a p such that $p^T H(\bar{x}) p < 0$. One possible direction is $p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Exercise 4.3.* *Write a* MATLAB *function newton.m that implements a modified Newton with a backtracking line search. (This is fairly simple modification of the routine steepest.m that you wrote for the previous homework.)*
Now, do the following with the implementation:

(a) *Starting at* $x_0 = (0, -1)^T$, *apply the modified Newton method to Rosenbrock's function*

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2,$$

which has a unique minimizer at $x^* = (1, 1)^T$.

(b) *Minimize the function*

$$f(x) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4,$$

starting at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$. *The minimizer lies at* $x^* = (0, 0, 0, 0)^T$. *Discuss the differences between this run and that of part (a).*

In each case, verify that the point you find is a local minimizer.

See the TA for the solution.