HW 5 Solutions

8.7.1

$$x'' = x$$
$$x(0) = x(1) = 1.$$

The general solution to the second order ODE $x'' = x$ is given by $x(t) = c_1 \cosh t + c_2 \sinh t$.

To find c_1, c_2 we plug in our initial conditions

$$1 = x(0) = c_1 \cosh 0 + c_2 \sinh 0. = c_1$$
$$1 = x(1) = c_1 \cosh 1 + c_2 \sinh 1$$

Thus. $1 = \cosh 1 + c_2 \sinh 1$ or.

$$c_2 = \frac{1 - \cosh 1}{\sinh 1}$$

Thus our solution is given by.

$$x(t) = \cosh t + \frac{1 - \cosh 1}{\sinh 1} \cdot \sinh t.$$

8.7.9

We have the two point boundary problem

$$x'' = f(x, t) = \cos(tx)$$
$$x(0) = 1 \quad x(1) = 4.$$

To convert this problem to one with homogeneous B.C we can use Thm 3. We let

$$h(p, q) = f(p, q + 1 + (4-1)p) = \cos(p(q + 3p + 1))$$

and solve the following problem

$$z'' = \cos(tz + 3t^2 + t)$$
$$z(0) = 0 \qquad z(1) = 0.$$

Then for z satisfying the modified problem, y, which solves our original problem is given by

$$y(t) = z(t) + 1 + 3t.$$

To show that our z-problem has a unique solution we need to ~~show~~. verify that $\cos(tz+3t^2+t)$ is continuous in $0 \le t \le 1$ $-\infty < s < \infty$. which it clearly is, moreover we must verify the Lipschitz condition i.e.

$$|\cos(tz_1+3t^2+t) - \cos(tz_2+3t^2+t)|$$

$$\le \cancel{K\cos(tz_1}$$

$$\le \left|\frac{\partial}{\partial z}\cos(tz+3t^2+t)\Big|_{z=z_3}\right| |z_1 - z_2|$$

by M.V.T. for some $z_3 \in [z_1, z_2]$.

$$\le |t\cos(tz_3+3t^2+t)| \cdot |z_1 - z_2|$$

$$\le t \cdot 1 |z_1 - z_2|.$$

$$\le 1|z_1 - z_2| \text{ since } t \in [0,1].$$

Thus we have a Lipschitz bound for $K = 1 < 8$ and by ~~...~~ Theorem 4, we have a unique solution for the modified problem, and hence we get a unique solution for the original problem.

8.9.3

We have the problem to find x a solution to.

$$x'' = u + v x + w x'$$
$$x(a) = \alpha \quad x(b) = \beta .$$

and to invoke Keller's Theorem we need to verify the three conditions on the top of page 592.

For the above problem we have

$$f(t, x, x') = u + v x + w x'$$

with $c_{11} = 1 \quad c_{12} = 0. \quad c_{13} = \alpha .$

$c_{21} = 1 \quad c_{22} = 0 \quad c_{23} = \beta$

(i) Need to verify f_t, f_x and $f_{x'}$ are continuous on $[a,b] \times \mathbb{R} \times \mathbb{R}$.

We have $f_t = 0.$

$$f_x = v.$$
$$f_{x'} = w$$

(Note we consider t, x and x' as independent variables when taking the above partial derivatives).

Assuming that u, v, w are all $C([a,b])$ we get that $f, f_t, f_x, f_{x'}$ are all continuous.

(ii) $f_x = v > 0$ by assumption and since $f_x = v$ and $f_{x'} = w$ are continuous on $[a,b]$ we have

$|v| \leq M$ and $|w| \leq M$ for some M

$= \max \left\{ \max_{x \in [a,b]} |v(x)|, \max_{x \in [a,b]} |w(x)| \right\}$

(iii) $|c_{11}| + |c_{12}| = 1 + 0 > 0$ ✓

$|c_{21}| + |c_{22}| = 1 + 0 > 0$ ✓.

$|c_{11}| + |c_{21}| = 1 + 1 > 0$ ✓.

$c_{11} c_{12} = 0 \leq 0$ ✓

$c_{21} c_{22} = 0 \geq 0$ ✓.

Since all the conditions hold our given problem

$$x'' = u + vx + wx'$$
$$x(a) = \alpha \quad x(b) = \beta$$

has a unique solution by Keller's Theorem.