8.4.1

The one step Adams's Moulton method is of the form

$$x_{n+1} = x_n + a f_{n+1} + b f_n.$$

which approximates the integral

$$\int_{t_n}^{t_{n+1}} f(t,x(t))dt = \frac{1}{h}(x(t_{n+1}) - x(t_n))$$

$$= \frac{1}{h}(x_{n+1} - x_n)$$

$$= \frac{a}{h} f_{n+1} + \frac{b}{h} f_n = \frac{a}{h} f(t_{n+1}, x_{n+1}) + \frac{b}{h} f(t_n, x_n)$$

To be exact for linear polynomials $h = t_{n+1} - t_n$.

$$p_0(t) = 1 \quad p_1(t) = t.$$

So.

$$t_{n+1} - t_n = \int_{t_n}^{t_{n+1}} 1\, dt = \frac{a}{t_{n+1}-t_n} \cdot 1 + \frac{b}{t_{n+1}-t_n} \cdot 1$$

$$\& \quad \frac{t_{n+1}^2}{2} - \frac{t_n^2}{2} = \int_{t_n}^{t_{n+1}} t\,dt = \frac{a}{t_{n+1}-t_n} \cdot t_{n+1} + \frac{b}{t_{n+1}-t_n} t_n.$$

or rather

$$h = \frac{a}{h} + \frac{b}{h}.$$

$$\frac{1}{2} h(t_{n+1} + t_n) = \frac{a}{h} t_{n+1} + \frac{b}{h} t_n.$$

$$\Rightarrow \quad a = b = \frac{1}{2} h^2.$$

$$\Rightarrow \quad \int_{t_n}^{t_{n+1}} f(t,x(t))dt \approx \frac{1}{2} h f(t_{n+1}, x_{n+1}) + \frac{1}{2} h f(t_n, x_n)$$

which is the Trap rule.

7.

<u>8.4.7</u> We are looking for a formula of the form.

$$x_{n+1} = x_n + a f_n + b f_{n-1}$$

which is exact for polynomials of order 1 and below. ie.

$$\int_{t_n}^{t_{n+1}} f(t, x(t))\,dt \approx h\left[A f_n + B f_{n-1}\right].$$

We assume $t_n = 0$ and $h = 1$.

and take as a basis for Π_2 us.

$$p_0(t) = 1.$$

$$p_1(t) = t.$$

$$\int_0^1 1\,dt = 1 = A + B \qquad \cancel{p_1(t)}\ p_1(-1).$$

$$\int_0^1 t\,dt = \frac{1}{2} = A \cdot 0 + B \cdot (-1)$$

So.

$$A + B = 1 \qquad \Rightarrow \cancel{A2} \quad B = -\frac{1}{2}$$

$$\cancel{A} - B = \frac{1}{2} \qquad\qquad A = \frac{3}{2}.$$

Giving us, since $\frac{a}{h} = A \quad \frac{b}{h} = B$

$$\Rightarrow. \quad x_{n+1} = x_n + h\left\{\frac{3}{2} f_n - \frac{1}{2} f_{n-1}\right\}. \quad \blacksquare$$

8.5.1

a) $x_n - x_{n-2} = 2h f_{n-1}$

So $p(z) = z^2 - 1$

$q(z) = 2z$.

Stable if roots of p lie in unit disk.

Consistent if $p(1) = 0$ $p'(1) = q(1)$.

Stable?. $z^2 - 1 = 0 \iff z = \pm 1 \Rightarrow$ Stable

Consistent? $p(1) = 1^2 - 1 = 0$ $p'(1) = 2 \cdot 1 = q(1) = 2 \cdot 1 \Rightarrow$ Consistent

Thus we have a convergent multistep method.

b) $x_n - x_{n-2} = h\left[\frac{7}{3} f_{n-1} - \frac{2}{3} f_{n-2} + \frac{1}{3} f_{n-3}\right]$.

~~$p(z) = z^2 - 1$~~ $q(z) = \frac{7}{3} z^2 - \frac{2}{3} z + \frac{1}{3}$.

$p(z) = z^3 - z$

Stable? $z^3 - z = 0 \iff z(z^2 - 1) = 0$

$z = 0$ is a ~~double~~ simple root. $\Rightarrow$ Stable!

$z = 1$ is a simple root

$z = -1$ is a simple root.

Consistent? $p(1) = 1^3 - 1 = 0$.

$$\left.\begin{aligned} p'(1) &= 3 \cdot 1^2 - 1 = 2. \\ q(1) &= \frac{7}{3} - \frac{2}{3} + \frac{1}{3} = \frac{6}{3} = 2 \end{aligned}\right\} \Rightarrow \text{Consistent}$$

$\Rightarrow$ b) is convergent.

9c) $X_n - X_{n-1} = h\left[\frac{3}{8}f_n + \frac{19}{24}f_{n-1} - \frac{5}{24}f_{n-2} + \frac{1}{24}f_{n-3}\right]$

$p(z) = z^3 - z^2 \qquad q(z) = \frac{3}{8}z^3 + \frac{19}{24}z^2 - \frac{5}{24}z + \frac{1}{24}.$

$p(z) = 0$

$z^3 - z^2 = 0. \iff z^2(z-1) = 0.$

$z=0$ is a double root
$z=1$ is a simple root $\Longrightarrow$ Stable.

~~$p'(z)$~~

$p(1) = 0 \qquad p'(1) = 3\,1^2 - 2\cdot 1 = 1.$

$q(1) = \frac{3}{8} + \frac{19}{24} - \frac{5}{24} + \frac{1}{24} = 1 \checkmark.$

Consistent.

Thus c) is convergent as well.

8.6.1

I did this in section twice, and Mike talked about it in class. The solution takes a lot of work, and it is not that important.

8.6.3

$$x''' + 2x'' - x' - 2x = e^t$$

$$x(8) = 3 \quad x'(8) = 2 \quad x(8) = 1$$

Let $y_0(t) = t$ which satisfies the ODE.

$$y_0'(t) = 1.$$

$$y_0(8) = 8.$$

Now let $y_1(t) = x(t)$.

$$y_2(t) = x'(t) = y_1'(t).$$

$$y_3(t) = x''(t) = y_2'(t)$$

and note $y_3'(t) = x'''(t)$ giving us.

$$\begin{bmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}' = \begin{bmatrix} 1 \\ y_2(t). \\ y_3(t) \\ -2y_3(t) + y_2(t) + 2y_1(t) + e^{y_0(t)} \end{bmatrix}$$

w/ initial conditions $\begin{bmatrix} y_0(8) \\ y_1(8) \\ y_2(8) \\ y_3(8) \end{bmatrix} = \begin{bmatrix} 8 \\ x(8) \\ x'(8) \\ x''(8) \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 2 \\ 1 \end{bmatrix}$