8.1.12

$$x' = 1 + x + x^2 \cos t \,. = f(x,t). \qquad (1)$$
$$x(0) = 0 \,.$$

For $(t,x) \in [-\frac{1}{3}, \frac{1}{3}] \times [-\beta, \beta]$

we notice that $|f(x,t)| \le 1 + |x| + |x^2||\cos t|$

$$\le 1 + \beta + \beta^2 \cdot 1 \,.$$

i.e. $|f(x,t)| \le M = 1+\beta+\beta^2$.

Let $R = \{(t,x) : |t| \le \frac{1}{3} \ \ |x| \le \beta\}$.

then by Theorem 8.1.1 (1) has a solution for $|t| \le \min\left(\frac{1}{3}, \frac{\beta}{1+\beta+\beta^2}\right)$. So the question of existence boils down to whether we can choose $\beta > 0$ s.t

$$\frac{\beta}{1+\beta+\beta^2} \ge \frac{1}{3}.$$

One can easily see that $\beta = 1$ satisfies the desired inequality. ■

8.1.18

$$x' = x^2$$
$$x(0) = 1.$$

Our rectangle is defined as.

$$R = \{(t,x) : |t| \le \alpha \quad \cancel{|x| \le \beta} . \; |x-1| \le \beta\}.$$

We need to find how large we can make the quantity

$$\min \{\alpha, \beta/M\}$$

where $M = \max_{(t,x)\in R} |f(x,t)|$. So we do some calculations. For ~~t~~ $|t| \le \alpha$ $|x-1| \le \beta$ we have.

$$x - 1 \le \beta \quad \text{and.} \quad 1 - x \le \beta.$$

or just $x \le \beta + 1$ (β is always positive).

then. $|f(x,t)| = x^2 \le (\beta+1)^2. = M.$

~~Now, how large can M be?~~.

$$\frac{\beta}{M} = \frac{\beta}{(\beta+1)^2}$$

and to maximize the interval of existence I want to choose $\beta > 0$ s.t $\frac{\beta}{(\beta+1)^2}$ is maximized. Turns out ~~this~~ by solving a simple calculus problem for maximizing $g(\beta) = \frac{\beta}{(\beta+1)^2}$ we get that $\max_{\beta>0} g(\beta) = \frac{1}{4}$. for $\beta = 1$.

Thus, if we take $\beta = 1$, $\frac{\beta}{(\beta+1)^2} = \frac{1}{4} = \cancel{\ } \beta/M$ and taking $\alpha = \frac{1}{4}$ we ~~get~~ insure existence on the interval $|t| \le \frac{1}{4}$.

Theorem 8.1.2 would predict the same interval of existence, but also insure uniqueness.

Theorem 8.1.3 requires f to be globally Lipschitz (i.e. with ~~~~ no condition on the size of x) which we cannot insure for $f(x,t) = x^2$.

Since x^2 is not a globally Lipschitz function.

The actual solution to our ODE is

$$x(t) = \frac{1}{1-t}$$

which ~~has a~~ is finite for any ~~interval~~ closed interval contained in $(-\infty, 1)$.

8.2.2

$$x' = x^{1/2}$$
$$x(0) = 0.$$

Since $\frac{d}{dt}\left(\frac{t^2}{4}\right) = \frac{t}{2} = \sqrt{\frac{t^2}{4}}$

and $\left.\frac{t^2}{4}\right|_{t=0} = 0$ we see $x(t) = \frac{t^2}{4}$

is an analytical solution to our ODE.

The first order Taylors method, ~~as~~ more commonly known Eulers method ~~is~~ for our problem is ~~$x_{k+1} =$~~

$$x(t+h) = x(t) + h\sqrt{x(t)}$$

Starting at $t=0$ $x(0)=0$ and we get.

$$x(h) = 0 + h\sqrt{0} = 0.$$

Then. $x(2h) = 0 + h\sqrt{0} = 0$ ~~which~~ any we see that we will always get zero from this method.

The key thing to note here is that our ODE does <u>not</u> have a unique solution, and Euler's method is picking out ~~one~~ the other solution $x(t)=0$.

<u>8.2.8</u> $x' = f(x,t)$.

$$x'' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} = f_t + f_x f.$$

$$x''' = f_{tt} + f_{tx}\cdot f + f_{xt} f + f_{xx}\cdot f^2 + f_x\cdot(f_t + f_x f).$$

$$= f_{tt} + 2 f_{xt}\cdot f + f_{xx} f^2 + f_x f_t + f_x^2 f$$

$$x'''' = f_{ttt} + f_{ttx}\cdot f + 2(f_{xtt} + f_{xtx} f)\cdot f$$
$$+ 2f_{xt}(f_t + f_x f) + (f_{xxt} + f_{xxx} f) f^2$$
$$+ f_{xx}\cdot 2f\cdot(f_t + f_x f) + \cancel{(f_{xxt} + f_{xxx} f) f^2}$$
$$+ (f_{xt} + f_{xx} f) f_t + f_x (f_{tt} + f_{tx} f)$$
$$+ 2f_x\cdot(f_{xt} + f_{xx} f) f + f_x^2 (f_t + f_x f).$$

(I think this is correct ...).

8.3.1

The second order Runge-Kutta method is

$$x(t+h) = x(t) + \omega_1 h f(x,t) + \omega_2 h f(x+\beta h f, t+\alpha h) + O(h^3).$$

Need
$$\omega_1 + \omega_2 = 1$$
$$\omega_2 \alpha = \tfrac{1}{2}$$
$$\omega_2 \beta = \tfrac{1}{2}.$$

Since $\alpha = \frac{2}{3}$ we get $\beta = \frac{2}{3}$ and $\omega_2 = \frac{3}{4}$

$\Rightarrow \omega_1 = \frac{1}{4}.$

Thus our formula is:

$$x(t+h) = x(t) + \frac{1}{4} h f(x,t) + \frac{3}{4} h f\left(x + \frac{2}{3} h f(x,t), \ldots t + \frac{2}{3} h\right).$$