~~HW 2~~

HW 2 Solutions

7.2.18 and 19.

I'm not sure what #19 wants different, so I'll just derive the composite rule for $\int_a^b f(x)\,dx$ for unequal spacing.

Let $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx.$$

Now to approximate $\int_{x_i}^{x_{i+1}} f(x)\,dx$ we transform to an integral over $[-1,1]$ and use the given midpoint rule.

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \int_{-1}^{1} f\left(x_i + \frac{(x_{i+1}-x_i)}{2}(y+1)\right) \cdot \frac{(x_{i+1}-x_i)}{2}\,dy.$$

$$y = 2\frac{x-x_i}{x_{i+1}-x_i} - 1$$

$$dy = \frac{2}{x_{i+1}-x_i}\,dx.$$

$$x = \frac{(x_{i+1}-x_i)}{2}(y+1) + x_i$$

$$\approx (x_{i+1}-x_i)\, f\left(x_i + \frac{(x_{i+1}-x_i)}{2}\right)$$

$$= (x_{i+1}-x_i)\, f\left(\frac{x_i + x_{i+1}}{2}\right)$$

Thus our composite rule for $\int_a^b f(x)dx$ is $\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} (x_{i+1}-x_i) f\left(\frac{x_i + x_{i+1}}{2}\right)$

If $x_i = a + \frac{b-a}{n} i$ (i.e equal spacing)

$$x_{i+1} - x_i = \frac{b-a}{n}$$

$$\frac{x_{i+1}+x_i}{2} = \frac{a + \frac{b-a}{n}(i+1) + a + \frac{b-a}{n} i}{2}$$

$$= a + \frac{b-a}{n}\left(i+\tfrac{1}{2}\right)$$

So the rule for equal spacing is:

$$\int_a^b f(x)dx \approx \cancel{\sum_{i=0}^{n-1}} \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n}\left(i+\tfrac{1}{2}\right)\right).$$

7.3.7

a) Let $(p,q) = \cancel{\int_0^1 x f(x) dx} = \int_0^1 x\, p(x) q(x) dx$

$n=1$, we want to find $q \in P_2$ which is orthogonal (w.r.t our above inner-product) to all polynomials in P_1. To do this, we perform Gram-Schmidt on $\{1, x, x^2\}$ w.r.t our inner-product.

Let $q_0(x) = 1$

$$q_1(x) = x - \frac{(x, q_0)}{(q_0, q_0)} q_0 .$$

$$q_2(x) = x^2 - \frac{(x^2, q_1)}{(q_1, q_1)} q_1 - \frac{(x^2, q_0)}{(q_0, q_0)} q_0$$

$$(x, q_0) = \int_0^1 x \cdot x \cdot 1 = \int_0^1 x^2 dx = \frac{1}{3}.$$

$$(q_0, q_0) = \int_0^1 x \cdot 1 \cdot 1 = \int_0^1 x\, dx = \frac{1}{2}.$$

$$\Rightarrow \quad q_1(x) = x - \frac{2}{3}.$$

$$(x^2, q_1) = \int_0^1 x \cdot x^2 (x - \tfrac{2}{3}) dx = \frac{1}{30}.$$

$$(q_1, q_1) = \int_0^1 x (x - \tfrac{2}{3})(x - \tfrac{2}{3}) dx = \frac{1}{36}.$$

$$(x^2, q_0) = \int_0^1 x \cdot x^2 \cdot 1 = \frac{1}{4}.$$

$$\Rightarrow q_2(x) = x^2 - \frac{36}{30}\left(x - \frac{2}{3}\right) - \frac{1}{2}$$

$$= x^2 - \frac{6}{5} x + \frac{3}{10}.$$

Now let $q = q_2$. Since then q is orthogonal to ~~q_0, q_1~~, q_0 and q_1 which span the polynomials $\mathcal{P}_1$, we have what we want.

Now we take our quadrature points to be the zeros of q which are

$$x_0 = \frac{3}{5} + \frac{\sqrt{6}}{10} \qquad x_1 = \frac{3}{5} - \frac{\sqrt{6}}{10}$$

Now we need to find A_0 and A_1 so that $\int_0^1 x f(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1)$ is exact for ~~a~~ all polynomials in $\mathcal{P}_3$.

To do so we use the method of undetermined coefficients.

$$\frac{1}{2} = \int_0^1 x \cdot 1\,dx = A_0 \cdot 1 + A_1 \cdot 1$$

$$\frac{1}{3} = \int_0^1 x \cdot x\,dx = A_0 x_0 + A_1 x_2 .$$

Solving (via MATLAB)

$$A_0 = \frac{1}{4} + \frac{\sqrt{6}}{36} \qquad A_1 = \frac{1}{4} - \frac{\sqrt{6}}{36} .$$

So our quadrature rule is

$$\int_0^1 x f(x)\,dx \approx \left(\frac{1}{4} + \frac{\sqrt{6}}{36}\right) f\left(\frac{3}{5} + \frac{\sqrt{6}}{10}\right) + \left(\frac{1}{4} - \frac{\sqrt{6}}{36}\right) f\left(\frac{3}{5} - \frac{\sqrt{6}}{10}\right)$$

which you can check to be ~~onto~~ exact for all polynomials of the form

$$p(x) = ax^3 + bx^2 + cx + d .$$

b. Since $n=2$ we need to find a third order polynomial which is orthogonal to $\{1, x, x^2\}$. Since the previous $\{q_0, q_1, q_2\}$ span $\mathcal{P}_2$ we just need to perform one more step of the Gram-Schmidt.

$$q_3(x) = x^3 - \frac{(x^3, q_2)}{(q_2, q_2)} q_2 - \frac{(x^3, q_1)}{(q_1, q_1)} q_1 - \frac{(x^3, q_0)}{(q_0, q_0)} q_0.$$

$$(x^3, q_2) = \int_0^1 x \cdot x^3 \cdot \left(x^2 - \frac{6}{5}x + \frac{3}{10}\right) dx = \frac{1}{350}.$$

$$(q_2, q_2) = \int_0^1 x \left(x^2 - \frac{6}{5}x + \frac{3}{10}\right)^2 dx = \frac{1}{600}.$$

$$(x^3, q_1) = \int_0^1 x \cdot x^3 \left(x - \frac{2}{3}\right) dx = \frac{1}{30}$$

$$(x^3, q_0) = \int_0^1 x \cdot x^3 \cdot 1 \, dx = \frac{1}{5}$$

$$q_3(x) = x^3 - \frac{600}{350}\left(x^2 - \frac{6}{5}x + \frac{3}{10}\right) - \frac{36}{30}\left(x - \frac{2}{3}\right) - \frac{2}{5}$$

$$= x^3 - \frac{12}{7}x^2 + \frac{6}{7}x - \frac{4}{35}$$

Let $q = q_3$ and now we find the zeros of q, (use MATLAB)

$x_0 \approx 0.911412$

$x_1 \approx 0.2123405$.

$x_2 \approx 0.590533$

Now we have the rule.

$$\int_0^1 x f(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

Plugging in $f(x) = 1$, x, and x^2 we find that

$A_0 \approx 0.2009319$

$A_1 \approx 0.0698269.$

$A_2 \approx 0.2292411$

7.2.20

Reproduce the steps for 7.2.7 but you don't have to find the zeros of q. Also, the new inner-product should be given by $(p,q) = \int_{-1}^{1} (1+x^2)p(x)q(x)\,dx$.

Find $\{q_0, q_1, q_2, q_3\}$ using Gram Schmidt on $\{1, x, x^2, x^3\}$. Take $q = q_3$.

7.4.7

$$S(f,h) - I = c_4 h^4 + c_6 h^6 + \dots$$

$$S(f,\tfrac{h}{3}) - I = c_4 \frac{h^4}{3^4} + c_6 \frac{h^6}{3^6} + \dots$$

$$-\; 3^4 S(f,\tfrac{h}{3}) - 3^4 I = c_4 h^4 + c_6 \frac{h^6}{3^2} + \dots$$

$$\cancel{S(f,h)}$$

$$S(f,h) - 3^4 S(f,\tfrac{h}{3}) - (1-3^4) I = \cancel{c_6(1}$$

$$= \frac{8}{9} c_6 h^6 + \dots$$

$$\Rightarrow \frac{1}{(1-3^4)}\left[S(f,h) - 3^4 S(f,\tfrac{h}{3}) \right] - I$$

$$= \frac{8 c_6}{9(1-3^4)} h^6 + \dots$$

Thus. if our new quadrature rule is given by. $\int_a^b f(x)\,dx = I \approx \frac{1}{(1-3^4)}\left[S(f,h) - 3^4 S(f,\tfrac{h}{3}) \right]$

it is $O(h^6)$ accurate.