HW 1 SOLUTIONS

6.1.22

$$c_k = \frac{y_k - p_{k-1}(x_k)}{(x_k - x_0)\cdots(x_k - x_{k-1})}$$

$$p_k(x) = \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1} (x - x_j)$$

$x_0 = -2 \quad y_0 = 0.$

$x_1 = 0 \quad y_1 = 1$

$x_2 = 1 \quad y_2 = -1$

$\cancel{p_0(x)}$

$$p_0(x) = c_0 = y_0 = 0.$$

$$p_1(x) = c_0 + c_1(x - x_0)$$

where $c_1 = \dfrac{y_1 - p_0(x_1)}{(x_1 - x_0)} = \dfrac{1 - 0}{2} = \dfrac{1}{2}.$

So $p_1(x) = \frac{1}{2}(x+2)$

$\frac{3}{2} - \frac{7}{2}$

$$p_2(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1)$$

$$c_2 = \frac{y_2 - p_1(x_2)}{(x_2 - x_0)(x_2 - x_1)} = \frac{-1 - \left(\frac{1}{2}\cdot(1+2)\right)}{3 \cdot 1} = \frac{-1 - \frac{3}{2}}{3} = \frac{-5}{6}.$$

$\frac{3}{2} - \frac{5}{6}\cdot 3 = \frac{-2}{2} = -1.$

$$p_2(x) = \frac{1}{2}(x+2) - \frac{5}{6}(x+2)x$$

$$= \frac{1}{2}x + 1 - \frac{5}{6}x^2 - \frac{5}{3}x$$

$$= -\frac{5}{6}x^2 - \frac{7}{6}x + 1$$

Lagrange Form

$$p(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x).$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-1)}{(-2)(-3)} = \frac{1}{6}x(x-1)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+2)(x-1)}{(2)(-1)} = -\frac{1}{2}(x+2)(x-1)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+2)x}{(3)(1)} = \frac{1}{3}x(x+2).$$

So,

$$p(x) = 1\cdot\left(-\tfrac{1}{2}(x+2)(x-1)\right) + (-1)\left(\tfrac{1}{3}x(x+2)\right).$$

$$= (x+2)\left[-\frac{1}{2}(x-1) - \frac{1}{3}x\right]$$

$$= (x+2)\left[-\frac{1}{2}x + \frac{1}{2} - \frac{1}{3}x\right] = (x+2)\left[\frac{1}{2} - \frac{5}{6}x\right].$$

$$= -\frac{5}{6}x^2 - \frac{7}{6}x + 1 \quad \checkmark$$

9 ~~7~~. 6.1.26

Let $y(x) = x - 9^{-x}$

$x_0 = 0 \qquad y_0 = 0 - 9^{-0} = -1$

$x_1 = 0.5 \qquad y_1 = \frac{1}{2} - 9^{-1/2} = \frac{1}{6}$

$x_2 = 1 \qquad y_2 = 1 - 9^{-1} = \frac{8}{9}.$

$$p(x) = -1\cdot\frac{(x-\frac{1}{2})(x-1)}{(0-\frac{1}{2})(0-1)} + \frac{1}{6}\,\frac{x(x-1)}{(\frac{1}{2}-0)(\frac{1}{2}-1)} + \frac{8}{9}\,\frac{x(x-\frac{1}{2})}{(1)(1-\frac{1}{2})}$$

$$= -\frac{8}{9}x^2 + \frac{25}{9}x - 1$$

Now solve $p(x) = 0$.

$$-\frac{8}{9}x^2 + \frac{25}{9}x - 1 = 0 \Rightarrow \quad x = \frac{-\frac{25}{9} \pm \sqrt{\left(\frac{25}{9}\right)^2 - 4\cdot\left(-\frac{8}{9}\right)\cdot(-1)}}{2\cdot\left(-\frac{8}{9}\right)}$$

$$\approx 0.41515 \text{ or } 2.70985$$

So we take 0.41515 to be an approximate solution in $[0,1]$ to $x - 9^{-x} = 0$.

7.1.14

We have the following Taylor series.

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \frac{f'''(\xi_{x,h})}{6}h^3$$

$$f(x+2h) = f(x) + 2f'(x)h + 2f''(x)h^2 + \frac{4f'''(\xi_{x,2h})}{3}h^3.$$

where $\xi_{x,h}$, $\xi_{x,2h}$ are numbers in some suff small neighborhood of x.

Now

$$4f(x+h) - f(x+2h) = 3f(x) + 2f'(x)h + \frac{2f'''(\xi_{x,h}) - 4f'''(\xi_{x,2h})}{3}h^3$$

$$-3f(x) + 4f(x+h) - f(x+2h)$$

$$= 2f'(x)h + \frac{2f'''(\xi_{x,h}) - 4f'''(\xi_{x,2h})}{3}h^3$$

Solving for $f'(x)$ we get.

$$f'(x) = \frac{1}{2h}\left[-3f(x) + 4f(x+h) - f(x+2h)\right]$$

$$- \left[\frac{f'''(\xi_{x,h}) - 2f'''(\xi_{x,2h})}{3}\right] h^2 .$$

Thus the error term in the approximation is

~~Ex.~~ 7.1.15

Let $\varphi(h) := \frac{1}{2h}\left[f(x+h) - f(x-h)\right]$.

Then we have. $f'(x) = \varphi(h) - \frac{h^2}{6} f'''(x) - \frac{h^4}{120} f^{(5)}(x) - \ldots$

But we also have.

$$f'(x) = \varphi(\tfrac{h}{2}) - \frac{h^2}{24} f'''(x) - \frac{h^4}{\cancel{480}\ 960} f^{(5)}(x) - \ldots$$

or

$$4f'(x) = 4\varphi(\tfrac{h}{2}) - \frac{h^2}{6} f'''(x) - \frac{h^4}{\cancel{480}\ 480} f^{(5)}(x) - \ldots$$

~~So~~. Subtracting we get.

$$f'(x) - 4f'(x) = \varphi(h) - 4\varphi(\tfrac{h}{2}) - \frac{h^4}{160} f^{(5)}(x) - \ldots$$

$$-3f'(x) = \varphi(h) - 4\varphi(\tfrac{h}{2}) - \frac{h^4}{160} f^{(5)}(x) - \ldots$$

$$f'(x) = \frac{4}{3}\varphi(\tfrac{h}{2}) - \frac{1}{3}\varphi(h) + \frac{h^4}{480} f^{(5)}(x) - \ldots$$

Thus if we approximate $f'(x)$ by

$$\frac{4}{3}\varphi\left(\frac{h}{2}\right) - \frac{1}{3}\varphi(h)$$

$$= \frac{4}{3}\cdot\frac{1}{2(h/2)}\left[f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right]$$

$$- \frac{1}{3}\frac{1}{2h}\left[f(x+h) - f(x-h)\right].$$

$$= \frac{1}{h}\left[\frac{4}{3}f\left(x+\frac{h}{2}\right) - \frac{4}{3}f\left(x-\frac{h}{2}\right) - \frac{1}{6}f(x+h) + \frac{1}{6}f(x-h)\right]$$

we get an error term that looks like

$$\frac{h^4}{480}f^{(5)}(\xi) \quad \text{for some } \xi.$$

7.2.10

Want $\int_0^1 f(x)\,dx = Af\left(\frac{1}{3}\right) + Bf\left(\frac{2}{3}\right)$.

Let $f(x) \approx f\left(\frac{1}{3}\right)\cdot\frac{\left(x-\frac{2}{3}\right)}{\left(\frac{1}{3}-\frac{2}{3}\right)} + f\left(\frac{2}{3}\right)\frac{\left(x-\frac{1}{3}\right)}{\left(\frac{2}{3}-\frac{1}{3}\right)}$

(Lagrange polynomial interpolation).

$$f(x) \approx (2-3x)f\left(\tfrac{1}{3}\right) + (3x-1)f\left(\tfrac{2}{3}\right).$$

$$\int_0^1 f(x)\,dx \approx f\left(\tfrac{1}{3}\right)\int_0^1 2-3x\,dx + f\left(\tfrac{2}{3}\right)\int_0^1 (3x-1)\,dx.$$

$$\boxed{= \frac{1}{2}f\left(\tfrac{1}{3}\right) + \frac{1}{2}f\left(\tfrac{2}{3}\right)}.$$

Now to transform over $[a,b]$.

$$\int_{x=a}^{x=b} f(x)\,dx = (b-a)\int_{y=0}^{y=1} f(a+(b-a)y)\,dy.$$

Let $y = \dfrac{x-a}{b-a}$

$dy = \dfrac{1}{b-a}\,dx$.

$x = a+(b-a)y$

$$\approx (b-a)\left[\tfrac{1}{2} f\left(a+(b-a)\tfrac{1}{3}\right) + \tfrac{1}{2} f\left(a+(b-a)\tfrac{2}{3}\right)\right]$$