4.

MT 2 Solutions

1. We say that an ODE is well-posed if we have the following three conditions satisfied

   (i) Existence
   (ii) Uniqueness
   (iii) Stability with respect to initial data

We have a theorem which says that for the ODE

$$y' = f(t,y) \qquad t \in [a,b]$$
$$y(a) = \alpha$$

~~the~~ it will be well-posed if f is Lipschitz in the strip $D = \{(t,y) : a \le t \le b,\ -\infty < y < \infty\}$.

~~This can~~ This Lipschitz condition can be verified by showing $\left| \frac{\partial f}{\partial y}(t,y) \right|$ is uniformly bounded.

For our problem

$$f(t,y) = 1 - 5y + e^t$$

$$\frac{\partial f}{\partial y} = -5 \implies \left| \frac{\partial f}{\partial y} \right| \le 5$$

Thus we get well-posedness.

2. $y''' = y'' + 5y' - 2y$

$y(0) = y'(0) = 0 \quad y''(0) = 1$

Let $x_1 = y$.

$x_2 = y' = x_1'$

$x_3 = y'' = x_2'$

Then we have.

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = x_3 + 5x_2 - 2x_1$$

with initial conditions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

3.

(a) Forward Euler w/ $h = 0.5$ and two time steps

$$y(\tfrac{1}{2}) = y(0) + \tfrac{1}{2} f(0, y(0)).$$

$$y(1) = y(\tfrac{1}{2}) + \tfrac{1}{2} f(\tfrac{1}{2}, y(\tfrac{1}{2}))$$

where $y(0) = 1$.

So $y(\tfrac{1}{2}) = 1 + \tfrac{1}{2} f(0,1) = 1 + \tfrac{1}{2}\{1 - 5 \cdot 1 + e^0\}$

$$= \cancel{1} = \frac{-1}{2}$$

$$y(1) = \frac{-1}{2} + \frac{1}{2} f(\tfrac{1}{2}, -\tfrac{1}{2}) = -\frac{1}{2} + \frac{1}{2}\{1 - 5(\tfrac{-1}{2}) + e^{1/2}\}$$

$$= \frac{1}{2}\{-1 + 1 + \frac{5}{2} + e^{1/2}].$$

$$= \frac{5}{4} + \frac{1}{2} e^{1/2} \longleftrightarrow \text{approx solution at } t = 1$$

(b) The truncation error is the difference between our approximation of the derivative and the actual.

That is,

$$\frac{y_{k+1} - y_k}{h} - y'(t_k) = \text{Truncation error}$$

(c) For Forward Euler we have by Taylor.

$$y(t_{k+1}) = y(t_k) + y'(t_k) h + \frac{y''(t_k)}{2} h^2 + O(h^3)$$

For Euler we have.

$$\frac{y_{k+1} - y_k}{h} - y'(t_k) = \overbrace{\frac{1}{2} y''(t_k) h^{\cancel{2}} + O(h^2)}^{\text{Truncation error.}}$$

$$\omega_0 = \alpha$$

4. $$\omega_{i+1} = \omega_i + h\left[\theta f(t_i, \omega_i) + (1-\theta) f(t_{i+1}, \omega_{i+1})\right]$$

(a) $\theta = 1$ gives us.

$$\omega_{i+1} = \omega_i + h f(t_i, \omega_i)$$

with the polynomials $p(z) = z - 1$.

$$q(z) = 1.$$

The root of p is $z = 1$ and thus the method is stable.

Since ~~q~~ $p(1) = 0$ and $p'(1) = 1 = q(1)$ the method is also consistent.

To ~~determine~~ determine the region of stability we analyze the ~~root~~ method for the ODE

$$y' = \lambda y \quad w/ \ \lambda < 1.$$

Then the one-step method is.

$$y_{k+1} = y_k + h\lambda y_k.$$

We want to find a range for h s.t $y_k \to 0$ as $k \to \infty$. To do so, we have the recursion formula.

$$y_k = (1 - h|\lambda|)^k y_0.$$

and in order for $y_k \to 0$ we need

$$|1 - h|\lambda|| < 1 \quad \text{or.} \quad -1 < 1 - h|\lambda| < 1$$

$\Rightarrow$. $h < \frac{2}{|\lambda|}$ ← This is our region of stability.

(b.) ~~In this~~ For $\theta = 0$ we have.

$$w_{i+1} = w_i + h f(t_{i+1}, w_{i+1})$$

giving us the polynomials $p(z) = z - 1$

$q(z) = z$.

One can then verify, the method is stable and consistent.

To get the region of stability we again examine the simple ODE

$$y' = \lambda y \quad \lambda < 0$$

and our one-step method gives us

$$y_{k+1} = y_k + h\lambda y_{k+1}$$

or. $$y_{k+1} = \frac{1}{(1 + h|\lambda|)^k} y_0$$

but since h and $|\lambda|$ are positive, clearly

$$y_k \to 0.$$

Thus for $\Theta = 0$ $(0, \infty)$ is our region of stability.