MT 1 Solutions (somewhat...)

- $P_2(x) = f(x_0)\, q_0(x) + f(x_1)\, q_1(x) + f(x_2)\, q_2(x)$

where $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2.$

$f(x_0) = 1 \quad f(x_1) = 2 \quad f(x_2) = 13.$

$$q_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{2}(x-1)(x-2)$$

$$q_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -x(x-2)$$

$$q_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{2}x(x-1).$$

- Plug $\frac{1}{2}$ into P_2 & to approximate f @ $x = 1/2$.
- The general form for the error is given by

$$f(x) - P_2(x) = \frac{1}{(2+1)!} f^{(3)}(\xi_x) \cdot x(x-1)(x-2)$$

where $\xi_x \in (a, b)$.

- If $f(x) = 2x^3 - x^2 + 1$ then

$$f^{(3)}(x) = 12$$

and.

$$|f(x) - P_2(x)| = \left|\frac{12}{6} \cdot x(x-1)(x-2)\right| \quad x \in [0,2].$$

$$\le 2 \cdot 2 \cdot 1 \cdot 2 = 8$$

(This is a bad estimate, but it works)

2. Taylor Expansion says.

$$f(x+h) = f(x) + f'(x)h + \tfrac{1}{2}f''(x)h^2 + \tfrac{1}{6}f'''(x)h^3 + o(h^4)$$

$$f(x+2h) = f(x) + 2f'(x)h + 2f''(x)h^2 + \tfrac{4}{3}f'''(x)h^3 + o(h^4)$$

Thus.

$$\begin{aligned} f(x) - 2f(x+h) + f(x+2h) &= f(x) - 2f(x) - 2f'(x)h - f''(x)h^2 - \tfrac{1}{3}f'''(x)h^3 \\ &\quad + f(x) + 2f'(x)h + 2f''(x)h^2 + \tfrac{4}{3}f'''(x)h^3 \\ &\quad + o(h^4). \end{aligned}$$

i.e.

$$\frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} = f''(x) + f'''(x)h + o(h^2).$$

Thus an expression for the error is given

by. $\boxed{f'''(\xi_x)h}$ for some $\xi_x \in [0, h]$.

- $f''(0) \approx \dfrac{f(0) - 2f(1) + f(2)}{1^2} = \dfrac{1 - 4 + 13}{1} = 10.$

3. Trapezoid Rule

$$\int_0^2 x^4\,dx \approx \left[\tfrac{1}{2}(0)^4 + \tfrac{1}{2}(2)^4\right]\cdot(2-0)$$

$$= 16.$$

Simpson's Rule

$$\int_0^2 x^4\,dx \approx \frac{2-0}{6}\left[(0)^4 + 4(1)^4 + (2)^4\right].$$

$$= \frac{1}{3}\left[4 + 16\right] = 20/3.$$

The error term for the Trapezoid rule is given by

$$-\frac{1}{12}(b-a)^3 f''(\xi) \qquad \xi \in (a,b).$$

for Simpson's rule.

$$-\frac{1}{90}\left[\frac{(b-a)}{2}\right]^5 f^{(4)}(\xi) \qquad \xi \in (a,b).$$

For our particular choice of $f(x) = x^4$ we have for the Trap rule;

$$-\frac{1}{12}\cdot 2^3 \cdot 12\,\xi^2 = -8\xi^2$$

i.e. $|\text{error}| \le 8\cdot 2^2 = 32.$ ← Error bound for trap.

for Simpson

$$-\frac{1}{90}\left[\frac{2}{2}\right]^5 f^{(4)}(\xi) = -\frac{1}{90}\cdot \cancel{16}\,24$$

i.e a bound for the error is

$$|\text{error}| \le 24/90.$$

Now, the exact integral is.

$$\int_0^2 x^4 dx = \frac{1}{5}x^5\Big|_0^2 = \frac{1}{5}\cdot 2^5 = \frac{32}{5}.$$

The approx using Trap rule was.

$$\left|16-\frac{32}{5}\right| < 32 \quad \checkmark.$$

using Simpson's rule.

$$\left|\frac{20}{3}-\frac{32}{5}\right| = \frac{24}{90}$$ ← Exactly the predicted error.

We want to now consider the composite rules which are given by.

$$\frac{-1}{12}(b-a)h^2 f''(\xi) \quad \text{and} \quad \frac{-1}{180}(b-a)h^4 f^{(4)}(\xi).$$

Want the composite error less ~~that~~ than 10^{-5}

That is.

$$\left|\frac{-1}{\cancel{12}_{\cancel{6}\,3}}\cdot\cancel{2}\,h^2\cdot\frac{\cancel{32}}{16}\right| \le 10^{-5}.$$

and $$\left|-\frac{1}{180}(2)h^4\cdot 24\right| \le 10^{-5}$$

or. $$\frac{16}{3}h^2 \le 10^{-5}$$

or. $$h \le \sqrt{\frac{3}{16}10^{-5}}$$

so. $$h \le \sqrt[4]{\frac{180}{48}\cdot 10^{-5}}$$

4. $$f'(x) = \left[\frac{f(x+h)-f(x)}{h}\right] - \frac{f''(x)}{2}h - \frac{f'''(x)}{6}h^2 + O(h^3)$$

is written as.

$$M = N(h) + k_1 h + k_2 h^2 + O(h^3).$$

~~We want to find a, b, c s.t if we mul~~

We also have

$$M = N(2h) + 2k_1 h + 4k_2 h^2 + O(h^3)$$
$$M = N(3h) + 3k_1 h + 9k_2 h^2 + O(h^3).$$

If we then multiply each equation by a, b, c and add we have.

$$(a+b+c)M = \cancel{(a+b} \; aN(h) + bN(2h) + cN(3h)$$
$$+ (a+2b+3c)k_1 h$$
$$+ (a+4b+9c)k_2 h^2$$
$$+ O(h^3).$$

We get what we want if the following hold

$$a+b+c=1$$
$$a+2b+3c=0.$$
$$a+4b+9c=0.$$

and then.

$$aN(h) + bN(2h) + cN(3h)$$ would be

our new approx with error $O(h^3)$.

Solving the above system for a, b, c we get

$$a = 3 \quad b = -3 \quad c = 1.$$