Math 170 B HW 5

1PM

6.4

#21 $s''(x) = \begin{cases} 6x & x \in [-1, \frac{1}{2}] \\ 18x & x \in [\frac{1}{2}, 1] \end{cases}$

$s''(-1) = 6 \cdot (-1) \neq 0 \Rightarrow$ Not a natural Cubic spline

#22 $\underbrace{s''(-1) \neq 0}_{\text{same as \#21}} \Rightarrow$ Not a Natural Cubic spline

6.7

#16 a) Differentiating:

$$+\sin(x) = -\sum_{k=1}^{\infty} \frac{(-1)^k (2k)\, x^{2k-1}}{(2k)!} = -\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k-1)!}$$

$$= +\sum_{k=0}^{\infty} \left((-1)^k x^{2k+1}\right) / \left((2k+1)!\right)$$

b) Integrating

$$\sin(x) = \sum_{k=0}^{\infty} \left(\frac{(-1)^k x^{2k+1}}{(2k+1)[(2k)!]} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

7.1

#6 (a) By Taylor series:

$$f(x \pm h) = f \pm hf' + \frac{h^2}{2} f'' \pm \frac{h^3}{6} f''' + \frac{h^4}{24} f^{(4)} \pm \frac{h^5}{120} f^{(5)} + \dots$$

$$f(x \pm 2h) = f \pm 2hf' + \frac{4h^2}{2} f'' \pm \frac{8h^3}{6} f''' + \frac{16h^4}{24} f^{(4)} \pm \frac{32h^5}{120} f^{(5)} + \dots$$

Thus: $f(x+h) - f(x-h) = 2hf' + \frac{2h^3}{6} f''' + \frac{2h^5}{120} f^{(5)} + \dots$ and

$$f(x+2h) - f(x-2h) = 4hf' + \frac{16h^3}{6} f''' + \frac{64h^5}{120} f^{(5)} \dots$$

Thus $\left(\frac{1}{12h}\right)\left[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)\right]$

$$= \frac{12h f'}{12h} + 0 \cdot \cancel{\frac{h^3 f'''}{6 \cdot 12h}} \to 0 - \frac{48h^5}{12h \cdot 120} f^{(5)} - \dots$$

$\Rightarrow$ error term $= \frac{h^4}{30} f^{(5)}(\xi) = O(h^4)$

(b) Same idea for: $f(x \mp h)$, $f(x \mp 2h)$, $f(x)$, and combine multiples

7.1 cont.

#12 $L = \phi(h) + a_1 h + a_3 h^3 + a_5 h^5 + \ldots$ (eq 1)

using $\frac{h}{2}$ $L = \phi\left(\frac{h}{2}\right) + a_1\left(\frac{h}{2}\right) + a_3\left(\frac{h}{2}\right)^3 + a_5\left(\frac{h}{2}\right)^5 + \ldots$ (eq 2)

$2 \cdot (\text{eq } 2) - (\text{eq } 1) \Rightarrow \boxed{L = 2\phi\left(\frac{h}{2}\right) - \phi(h) - \frac{3}{4} a_3 h^3 + \ldots}$

#14 Taylor series

$f(x+h) = f + hf' + \frac{h^2}{2} f'' + \frac{h^3}{6} f''' + \ldots$

$f(x+2h) = \ldots$

use given equation and solve for $f'(x)$ to get error term.

#16 same as #14, but solve for $f''(x)$ to get error term.

$\boxed{7.2}$ #10 $l_0(x) = \dfrac{\frac{2}{3} - x}{\frac{2}{3} - \frac{1}{3}} = 2 - 3x : \int_0^1 l_0(x)\,dx = A$

$l_1(x) = \dfrac{x - \frac{1}{3}}{\frac{2}{3} - \frac{1}{3}} = 3x - 1 : \int_0^1 l_1(x)\,dx = B$

$\Rightarrow A = B = \frac{1}{2}$

for general $[a, b]$

let $x = \lambda(t) = (b-a)t + a$, $dx = (b-a)dt$

$\Rightarrow \int_a^b f(x)\,dx = (b-a)\int_0^1 f(\lambda(t))\,dt$

$f \circ \lambda$ is now just composite function, plug into first part of question.

$\boxed{7.2}$ cont...

#11 $p(x) = f(x_1) + f[x_1, x_2](x - x_1) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$

$\Rightarrow \int_{x_0}^{x_3} p(x)\,dx = (x_3 - x_0) f(x_1) + \frac{f[x_1, x_2](x - x_1)^2}{2}\Big|_{x_0}^{x_3}$

--- Then write out and simplify.

#16 Assume $|f^{(n+1)}(x)| < M$ on $[0, 1]$

By T.2 section 6.1:

$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$

$\Rightarrow \int_0^1 f(x)\,dx = \int_0^1 p(x)\,dx + \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)\,dx$

$\Rightarrow |\text{Error}| < \frac{M}{(n+1)!} \int_0^1 \prod_{i=0}^{n} |x - x_i|\,dx$

use eq (15) to minimize product. (change of variables on interval gives another factor of 2)

#18 $\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \sum_{i=1}^{n} (x_i - x_{i-1}) f\left(\frac{x_i + x_{i-1}}{2}\right)$

If spacing is uniform:

$x_i = a + ih \; ; \; h = \frac{b-a}{n}$

$\Rightarrow \int_a^b f(x)\,dx \simeq h \sum_{i=1}^{n} f\left(a + \frac{(2i-1)h}{2}\right)$

#31 with trapezoid rule, error-term is $-\frac{1}{12}(b-a)h^2 f''(\xi)$

$f(x) = x + e^{-x^2}$ $\quad f' = 1 - 2x e^{-x^2}$

$f'' = -2(1 - 2x^2)e^{-x^2}$ $\quad f''' = 4x(3 - 2x^2)e^{-x^2}$

set f''' to zero and check B.C.'s to maximize $|f''|$

$\Rightarrow$ error $< \frac{1}{12} h^2 (\max f''(\xi))$; set less than $\frac{10^{-7}}{2}$

and solve for h. $n = \frac{1}{h}$