MATH 170B: Introduction to Numerical Analysis: Approximation and Nonlinear Equations

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Homework Assignment #3
Due (See Class Webpage for Due Date)

The starred exercises are those that require the use of MATLAB. *Remember, you must do the* MATLAB *problems to get credit for the homework.*

Exercise 3.1.* Compute the gradient $g(x) = \nabla f(x) = f'(x)^T$, and Hessian $H(x) = f''(x)$, of the following function:

$$f(x) = e^{x_3} x_1^2 + 2x_2^2 + x_3^2 \cos x_1.$$

Now write a MATLAB function with specification `[f,g,H] = ex31(x)` that computes $f(x)$, $g(x)$ and $H(x)$ for the function at any point x that is provided. Use your function to compute $f(x)$, $g(x)$, and $H(x)$ at $x = (0,\ 0,\ 0)^T$ and $x = (-1,\ 2,\ -2)^T$. In each case, use MATLAB to compute the spectral decomposition of the Hessian matrix, and indicate if the first and second order necessary conditions, and the second order necessary and sufficient conditions, for unconstrained local minimization are satisfied.

Exercise 3.2. Given each of the following cases of a gradient $g(\bar{x})$ and Hessian $H(\bar{x})$ defined at a point $\bar{x}$, discuss the optimality of $\bar{x}$. (Do NOT use MATLAB. You may need to know how to compute eigenvalues by hand in your next exam. Note that the 3×3 example requires only computing the eigenvalues of the 2×2 submatrix; in other words, for this problem all of the eigenvalues in each example are available by simply reading them off, or in the worst case, by using the quadratic formula.)

(i) $g(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$

(ii) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}.$

(iii) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}.$

(iv) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$

(v) $g(\bar{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad H(\bar{x}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$

Exercise 3.3. Let $q(x)$, $x \in \Re^n$, be the quadratic function $q(x) = c^T x + \frac{1}{2} x^T H x$, where H is symmetric.

(a) Write down an expression for $\nabla q(x)$ in terms of c, H and x.

(b) Given an arbitrary point x_0 and a direction p, write down the Taylor-series expansion of $q(x_0 + p)$.

(c) For this part, consider $q(x)$ such that H is positive definite. If p is a direction such that $\nabla q(x_0)^T p < 0$, show that there exists a *positive* minimizer α^* of $q(x_0 + \alpha p)$. Derive a closed-form expression for α^*.

Exercise 3.4.* Write a MATLAB m-file `newton.m` that implements Newton's method with backtracking line search (damping). Terminate the algorithm when either $\|g(x_k)\| \leq 10^{-5}$ or 75 iterations are performed.

Use `newton.m` to find a minimizer of the function

$$f(x) = e^{x_3} x_1^2 + 2x_2^2 + x_3^2 \cos x_1,$$

starting at the point $(-1,\ 1,\ 1)^T$. Note that as part of Exercise 3.1, you have already written a routine that will return $f(x)$, $g(x)$, and $H(x)$ at any given point $x = (x_1, x_2, x_3)$, corresponding to this definition of $f(x)$.

Exercise 3.5. Let $f(x)$ denote a convex continuously differentiable function. Show that if a stationary point x^* exists, then $f(x^*)$ is a global minimum of f. Also show that if $f(x)$ is actually strictly convex, then x^* is the unique global minimum. Why can uniqueness be lost if the function is not strictly convex? Draw a picture of such a situation when $f\colon \mathbb{R} \to \mathbb{R}$.

Hint: Use the result that $f(x)$ is convex if and only if $f(y) \geq f(x) + f'(x)(y - x)$ for all x and y, and the result that $f(x)$ is strictly convex if and only if the inequality holds strictly.