# AMa105

## Homework #3 (Approximation Theory)

**Handed out: 14 February 1996**
**Due in class: 28 February 1996**

Assignment: Complete any **three** of the following seven problems.

- **Problem 1.** Consider the problem of best approximation of the continuous function $f(x)$ over the interval $[0,1]$, where we are interested in the best approximation in the $L^2$-norm:

$$\|f\| = \left( \int_0^1 f^2 dx \right)^{1/2}.$$

Determine the best approximation to $f(x) = e^x$ by a constant function. (Note that this function is unique.) What is the resulting norm of the error in the approximation?

- **Problem 2.** If we equip a linear (vector) space $\mathcal{H}$ with a bilinear form $(\cdot,\cdot)$ satisfying the inner-product axioms, then we know that together $\{\mathcal{H},(\cdot,\cdot)\}$ form an inner-product space. Show that $\mathcal{H}$ is also a normed space. I.e., show that the functional defined by

$$\|u\| = (u,u)^{1/2}, \qquad \forall u \in \mathcal{H},$$

satisfies the norm axioms as given in class.

Hint: All of the properties are immediate except for the triangle inequality, which is non-trivial. You can assume that you have the Cauchy-Schwarz inequality at your disposal:

$$|(u,v)| \leq \|u\| \, \|v\|, \qquad \forall u,v \in \mathcal{H}.$$

- **Problem 3.** The Existence and Uniqueness Theorems for a best polynomial approximation to a continuous function were given in class for a "semi"-norm on the space of continuous functions. As a result, a technical assumption was made on the semi-norm of the form: There exists positive numbers $m_n$ and $M_n$ such that

$$0 < m_n \leq ||| \sum_{j=0}^{n} b_j x^j ||| \leq M_n, \qquad n = 0,1,\ldots,$$

for all $\{b_j\}$ satisfying a normalization condition given in class. Show that if the semi-norm $|||\cdot|||$ is strengthened to a norm, then the lower inequality involving $m_n$ always holds.

Hint: There is a hint in the book of Isaacson and Keller.

- **Problem 4.** Generalize the Existence Theorem given in class for best polynomial approximation of a continuous function to the more general setting of best approximation in a finite-dimensional subspace of a general normed space.

  Hint: A proof can be given which follows (very) closely the one given in class for polynomials.

- **Problem 5.** Generalize the Uniqueness Theorem given in class for best polynomial approximation of a continuous function to the more general setting of best approximation in a finite-dimensional subspace of a general normed space. You will need to assume that the finite-dimensional subspace forms a convex set.

  Hint: This one is a bit difficult; however, it illustrates how general some of these underlying ideas are.

- **Problem 6.** State and prove a tensor-product version of the Weierstrass Approximation Theorem for the uniform approximation of a continuous function $f(x,y)$, where $(x,y) \in [0,1] \times [0,1]$. Following the constructive approach given in class for the case of the interval $[0,1]$, you will need to employ the generalized Bernstein polynomials on $[0,1] \times [0,1]$ of the form:

  $$B_{m,n}(f;x,y) = \sum_{j=0}^{m} \sum_{k=0}^{n} f\left(\frac{j}{m}, \frac{k}{n}\right) \beta_{m,j}(x) \beta_{n,k}(y).$$

  Hint: There is a hint in the book of Isaacson and Keller.

- **Problem 7.** As presented in class, the Chebyshev polynomials are defined as:

  $$t_n(x) = \cos(n \cos^{-1} x), \qquad n = 0, 1, 2, \dots.$$

  Taking $t_0(x) = 1$, $t_1(x) = x$, we also noted that the Chebyshev polynomials can be generated by the recursion:

  $$t_{n+1}(x) = 2t_1(x)t_n(x) - t_{n-1}(x), \qquad n = 1, 2, 3, \dots.$$

  Prove the following extremely useful relationships:

  $$t_k(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^k + \left(x - \sqrt{x^2-1}\right)^k\right], \qquad \forall x, \tag{1}$$

  $$t_k\left(\frac{\alpha+1}{\alpha-1}\right) > \frac{1}{2}\left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}\right)^k, \qquad \forall \alpha > 1. \tag{2}$$

  (These two results are fundamental in the convergence analysis of the conjugate gradient iteration for solving linear systems.)

  Hint: For the first result, use the face that $\cos k\theta = (e^{ik\theta} + e^{-ik\theta})/2$. The second result will follow from the first after a lot of algebra.