AMa104

Homework #2

(Vector Spaces, Normed and Inner-product spaces, Orthogonality, and Linear Transformations)

Handed out: 22 October 1996
Due in class: 31 October 1996

- **Problem 1.** (Strang 2.1.5) Recall that a vector space V is a collection $V =< V, K, \cdot, + >$ which satisfies the ten rules V1-V10 given in class (Strang combines some of them to get eight rules).

  1. Taking $V = \mathbb{R}^2$, $K = \mathbb{R}$, suppose that addition in $\mathbb{R}^2$ adds an extra one to each component, so that $(3,1) + (5,0) = (9,2)$ rather than $(8,1)$. With scalar multiplication unchanged, which vector space rules are broken?
  2. Taking $V = \mathbb{R}$, $K = \mathbb{R}$, show that the set of all positive real numbers, with $x + y$ and $c \cdot x$ redefined to the usual xy and x^c, respectively, is a vector space. What is the "zero" vector in this new space?

- **Problem 2.** (Strang 2.3.1) Decide (rigorously) whether or not the following vectors are (1) linearly independent and (2) span $\mathbb{R}^4$.

$$v^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad v^4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

- **Problem 3.** (Strang 2.4.3) Find the dimension and a basis of the four fundamental subspaces for both

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- **Problem 4.** Let $V = \text{span}\{v^1, v^2\} \subseteq \mathbb{R}^2$, and let $W = \text{span}\{w^1, w^2\} \subseteq \mathbb{R}^3$, where

$$v^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad w^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad w^2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Construct the matrix representation A of the linear transformation which maps v^1 to w^1, and maps v^2 to w^2. If we change the basis for V to

$$v^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v^2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$

how does the matrix A change?

- **Problem 5.** (Strang 2.6.8) The space $\mathcal{P}_3$ of cubic polynomials is a vector space, and a standard basis is $\{1, t, t^2, t^3\}$. We know that differentiation and integration are "linear" transformations to functions, so we should be able to construct the matrix representation of these operations on $\mathcal{P}_3$ in this standard basis.

  1. Construct the matrix A representing d^2/dt^2.
  2. What are the null and range spaces of A?
  3. What is the interpretation of these spaces in terms of polynomials?

- **Problem 6.** Is the following system uniquely solvable? (If so, determine the solution)?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}.$$

  What if the right-hand side is changed to $(1, 4, 3)^T$? If it is solvable now, determine a solution. If the solution is not unique, determine a second solution. If you didn't approach this problem using the Fredholm alternative presented in class, then re-work the problem using the Fredholm alternative argument.

- **Problem 7.** Show that the function $d(u, v) : \mathbb{R}^n \to \mathbb{R}$, defined by

$$d(u, v) = \|u - v\|_{l^2} = \left(\sum_{i=1}^{n} |u_i - v_i|^2 \right)^{1/2}$$

  (where $\| \cdot \|_{l^2}$ is the usual Euclidean 2-norm) satisfies the three properties of a metric, and therefore gives $\mathbb{R}^n$ the topological structure of a metric space.

- **Problem 8.** In class, we connected vectors and linear operators on vector spaces with linear functionals and bilinear forms (respectively) through the Riesz representation theorem and the bounded operator theorem. Using these ideas, give a completely rigorous argument to conclude that if $A \in L(H, H)$ (i.e., A is a bounded linear operator on a finite-dimensional Hilbert space H), then the problem:

$$\text{Find } u \in H \text{ such that } Au = F \in H$$

  is equivalent to the problem

$$\text{Find } u \in H \text{ such that} \quad a(u, v) = f(v), \quad \forall v \in H,$$

  where A is related to the bilinear form $a(\cdot, \cdot)$ through the bounded operator theorem, and F is related to the linear form $f(\cdot)$ through the Riesz theorem. (For example, this allows us to replace a matrix equation with an equivalent equation involving linear and bilinear functionals when convenient.)

- **Problem 9.** In class a projection operator P was defined in terms of a direct sum, and as a consequence we saw that a linear operator P was a projection iff it was idempotent, i.e., $P^2 = P$. Prove that a linear operator P is a projection iff the operator $I - P$ is a projection, where I is the identity operator.