

A NOTE ON DETERMINING PROJECTIONS FOR NON-HOMOGENEOUS INCOMPRESSIBLE FLUIDS

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ABSTRACT. In this note, we consider a viscous incompressible fluid in a finite domain in both two and three dimensions, and examine the question of determining degrees of freedom (projections, functionals, and nodes). Our particular interest is the case of non-constant viscosity, representing either a fluid with viscosity that changes over time (such as an oil that loses viscosity as it degrades), or a fluid with viscosity varying spatially (as in the case of two-phase or multi-phase fluid models). Our goal is to apply the determining projection framework developed by the second author in previous work for weak solutions to the Navier-Stokes equations, in order to establish bounds on the number of determining functionals for this case, or equivalently, the dimension of a determining set, based on the approximation properties of an underlying determining projection. The results for the case of time-varying viscosity mirror those for weak solutions established in earlier work for constant viscosity. The case of space-varying viscosity, treated within a single-fluid Navier-Stokes model, is quite challenging to analyze, but we explore some preliminary ideas for understanding this case.

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1. INTRODUCTION

In the following, we consider a viscous incompressible fluid in $\Omega \subset \mathbb{R}^d$, where Ω is an open bounded domain with Lipschitz continuous boundary, and where $d = 2$ or $d = 3$. Given the kinematic viscosity $\nu > 0$, and the vector volume force function $f(x, t)$ for each $x \in \Omega$ and $t \in (0, \infty)$, the governing Navier-Stokes equations (NSE) for the fluid velocity vector $u = u(x, t)$ and the scalar pressure field $p = p(x, t)$ are given by:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \text{ in } \Omega \times (0, \infty), \quad (1.1)$$

$$\nabla \cdot u = 0 \text{ in } \Omega \times (0, \infty). \quad (1.2)$$

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One is also provided with initial conditions $u(0) = u_0$, as well as boundary conditions on $\partial\Omega \times (0, \infty)$. Our goal in this article is to examine some questions about a concept known as *determining degrees of freedom* in the flow described by (1.1)–(1.2). While the classical setting involves the assumption of a constant bulk viscosity $\nu > 0$, our particular interest here is in the case of non-constant viscosity, representing either a fluid with viscosity $\nu(t)$ that changes over time (such as an oil that loses viscosity as it degrades), or a fluid with viscosity $\nu(x)$ varying spatially (as in the case of two-phase or multi-phase fluid models), or both, represented by a viscosity $\nu(x, t)$ that changes over space and time. We will assume that ν is everywhere positive as a function of time and/or space, and will also assume that it *a priori* satisfies some uniform pointwise upper and lower bounds, based on some underlying physical considerations. Although we do not consider dependence of the viscosity on fluid velocity in this work, we note that there has long been an active numerical simulation community that studies this case, and there is now also growing interest in the analysis of the Navier-Stokes equations with variable viscosity; cf. [20, 19, 7, 6, 18, 8].

The notion of *determining modes* for the NSE was first introduced in [10] as an attempt to identify and estimate the number of degrees of freedom in turbulent flows; a thorough discussion of the role of determining sets in turbulence theory can be found in [5]. This core idea later led to the study of *Inertial Manifolds* [11]. Estimates of the number of determining modes under various assumptions have been developed since the early 1980's; some examples include [9, 26]. The notion of *determining nodes* and related concepts were introduced in [12, 13], followed by *determining volumes* in [14, 24], and various estimates of their number in different modeling scenarios can be found in e.g. [25, 26]. A unified framework for modes, nodes, and volumes was presented in [2, 3], including the relationship to Inertial Manifolds. In [21, 23], we extended the results of [2, 3] to the more general setting of weak solutions lying in a suitably defined divergence-free solution space V (see §2 below). In particular, we showed that if a *projection operator* $R_N: V \rightarrow V_N \subset L^2(\Omega)$ into a subset V_N with $N = \dim(V_N) < \infty$, satisfies an approximation inequality for $\gamma > 0$ of the form,

$$\|u - R_N u\|_{L^2(\Omega)} \leq C_1 N^{-\gamma} \|u\|_{H^1(\Omega)}, \quad (1.3)$$

then the operator R_N is a *determining projection* for the system (1.1)–(1.2) in the sense of Definition 1.1 below, provided N is large enough. Furthermore, in [21, 23], we also derived explicit bounds on the dimension N which guarantees that R_N is determining, and we gave explicit constructions of determining projections for both smooth and weak solutions using “rough” finite element quasi-interpolants. Our more recent article [22] generalized these results further, to a more general family of regularized NSE and MHD models that includes (1.1)–(1.2) as a special case. This area of research has continued to generate substantial activity; a survey through 2009 appears in our earlier article [22], and much more recent related activity includes [16, 34, 15] and the references therein, among many other related works that are too numerous to list here.

Bounds on the number of determining modes, nodes, and volumes are usually phrased in terms of a generalized *Grashof number*, defined for the two-dimensional NSE as:

$$Gr = \frac{c_\rho^2 F}{\nu^2} = \frac{F}{\lambda_1 \nu^2},$$

where λ_1 is the smallest eigenvalue of the Stokes operator and $c_\rho = 1/\sqrt{\lambda_1}$ is the related (best) Poincaré constant. Here, $F = \limsup_{t \rightarrow \infty} (\int_\Omega |f(x, t)|^2)^{1/2}$ if $f \in L^2(\Omega)$ for almost every t , or $F = \limsup_{t \rightarrow \infty} \sqrt{\lambda_1} \|f\|_{H^{-1}(\Omega)}$ if $f \in H^{-1}(\Omega)$ for almost every t . Due

to the failure of the Sobolev embedding $H^1 \hookrightarrow C^0$ in dimensions 2 and 3, determining node analysis, which was based on point-wise interpolants of the velocity, was limited to H^2 -regular solutions, although it was understood that determining modes and volume elements made sense under weaker conditions. To construct a general analysis framework for the case of weak (e.g., H^1 -regular solutions) solutions to (1.1)–(1.2), in [23] we introduced the notions of *determining projections* and *determining functionals*, which we now define. (The standard spaces H , V , and V' for (1.1)–(1.2) are reviewed below in §2.)

Definition 1.1 (Determining Projections for the NSE). *Let $f(t), g(t) \in V'$ be any two forcing functions satisfying*

$$\lim_{t \rightarrow \infty} \|f(t) - g(t)\|_{V'} = 0, \quad (1.4)$$

and let $u, v \in V$ be corresponding weak solutions to (1.1)–(1.2). The projection operator $R_N: V \rightarrow V_N \subset L^2(\Omega)$, $N = \dim(V_N) < \infty$, is called a **determining projection** for weak solutions of the d -dimensional NSE if

$$\lim_{t \rightarrow \infty} \|R_N(u(t) - v(t))\|_{L^2(\Omega)} = 0, \quad (1.5)$$

implies that

$$\lim_{t \rightarrow \infty} \|u(t) - v(t)\|_H = 0. \quad (1.6)$$

Given a basis $\{\phi_i\}_{i=1}^N$ for the finite-dimensional space V_N , and a set of bounded linear functionals $\{l_i\}_{i=1}^N$ from V' , a projection operator can be constructed as:

$$R_N u = \sum_{i=1}^N l_i(u) \phi_i. \quad (1.7)$$

Condition (1.5) is then implied by:

$$\lim_{t \rightarrow \infty} |l_i(u(t) - v(t))| = 0, \quad i = 1, \dots, N \quad (1.8)$$

and in this case we refer to $\{l_i\}_{i=1}^N$ as a set of **determining functionals**.

The analysis of whether R_N or $\{l_i\}_{i=1}^N$ are determining can be reduced to an analysis of the approximation properties of R_N . Note that in this construction, the basis $\{\phi_i\}_{i=1}^N$ need not span a subspace of the solution space V , so that the functions ϕ_i need not, for example, be divergence-free. Note that Definition 1.1 encompasses each of the notions of determining modes, nodes, and volumes by making particular choices for the sets of functions $\{\phi_i\}_{i=1}^N$ and $\{l_i\}_{i=1}^N$.

Outline. Preliminary material is presented in §2, including notation used for Lebesgue and Sobolev spaces and norms, and some inequalities for bounding the terms appearing in weak formulations of the NSE. In §3, we give an overview of the general framework for constructing determining projections for the NSE for both two and three spatial dimensions. To make use of the framework to establish bounds on the number of determining degrees of freedom for weak solutions, one must assume, or establish, a single *a priori* bound for solutions to the equations, and also provide a projection operator that satisfies a single approximation inequality. The remainder of the article then turns to the necessary *a priori* bounds for non-constant viscosity. In §4, we derive some *a priori* bounds for the case of time-varying viscosity that are needed to make use of the determining projection framework in §3. Section §5 looks briefly at a simplified model for space-varying viscosity. We first develop a natural weak formulation for a simplified model, where the viscosity is allowed to now be space-varying, but is also assumed to be

explicitly given as data, and in particular, does not depend on the fluid velocity. Using this simplified formulation, we then establish some basic *a priori* bounds for use with the determining projection framework from §3. Some additional technical tools are summarized in Appendix A, *a priori* estimates for the constant viscosity case are presented in Appendix B.

2. PRELIMINARY MATERIAL

We briefly review some background material and notation, following the approach taken in our earlier articles [21, 23, 22], which in turn followed the notational conventions used in [4, 29, 32, 33]. To keep the discussion in this section as clear and concise as possible, we have placed some technical results that are repeatedly used throughout the paper in Appendix A.

Let $\Omega \subset \mathbb{R}^d$ denote an open bounded set. The embedding and other standard results we will need to rely on are known to hold for example if the domain Ω has a locally Lipschitz boundary, denoted as $\Omega \in \mathcal{C}^{0,1}$ (cf. [1]). For example, open bounded convex sets $\Omega \subset \mathbb{R}^d$ satisfy $\Omega \in \mathcal{C}^{0,1}$ (Corollary 1.2.2.3 in [17]), so that convex polyhedral domains are in $\mathcal{C}^{0,1}$. Let $H^k(\Omega)$ denote the usual Sobolev spaces $W^{k,2}(\Omega)$. Employing multi-index notation, the distributional partial derivative of order $|\alpha|$ is denoted D^α , so that the (integer-order) norms and semi-norms in $H^k(\Omega)$ may be denoted

$$\|u\|_{H^k(\Omega)}^2 = \sum_{j=0}^k |\Omega|^{\frac{j-k}{d}} |u|_{H^j(\Omega)}^2, \quad |u|_{H^j(\Omega)}^2 = \sum_{|\alpha|=j} \|D^\alpha u\|_{L^2(\Omega)}^2, \quad 0 \leq j \leq k,$$

where $|\Omega|$ represents the measure of Ω . Fractional order Sobolev spaces and norms may be defined for example through Fourier transform and extension theorems, or through interpolation. A fundamentally important subspace is the $k = 1$ case of

$$H_0^k(\Omega) = \{\text{closure of } C_0^\infty(\Omega) \text{ in } H^k(\Omega)\},$$

for which the Poincaré Inequality holds. (See Lemma A.2 in Appendix A.) The spaces above extend naturally (cf. [32]) to product spaces of vector functions $u = (u_1, u_2, \dots, u_d)$, which are denoted with the same letters but in bold-face; for example, $\mathbf{H}_0^k(\Omega) = (H_0^k(\Omega))^d$. The inner-products and norms in these product spaces are extended in the natural Euclidean way; the convention here will be to subscript these extended vector norms the same as the scalar case.

Define now the space \mathcal{V} of divergence free C^∞ vectors with compact support as

$$\mathcal{V} = \{\phi \in C_0^\infty(\Omega) \mid \nabla \cdot \phi = 0\}.$$

Two subspaces of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ are fundamental to the study of the NSE:

$$H = \{\text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega)\}, \quad V = \{\text{closure of } \mathcal{V} \text{ in } \mathbf{H}_0^1(\Omega)\}.$$

We use a fairly standard notation (cf. [32]) for inner-products and norms in H and V is:

$$(u, v)_H = (u, v)_{L^2(\Omega)}, \quad \|u\|_H = \|u\|_{L^2(\Omega)}, \quad (2.1)$$

$$(u, v)_V = [u, v]_{H^1(\Omega)}, \quad \|u\|_V = |u|_{H^1(\Omega)}. \quad (2.2)$$

Thanks to the Poincaré inequality, the H^1 -semi-inner-product $[u, v]_{H^1(\Omega)}$ is an inner-product on V , and the H^1 -semi-norm $|u|_{H^1(\Omega)}$ is a norm on V .

The NSE (1.1)–(1.2) with homogeneous Dirichlet (no-slip) boundary conditions are equivalent (cf. [32]) to the functional differential equation:

$$\frac{du}{dt} + \nu Au + B(u, u) = f, \quad u(0) = u_0, \quad (2.3)$$

where the Stokes operator A and bilinear map B are defined as

$$Au = -P\Delta u, \quad B(u, v) = P[(u \cdot \nabla)v],$$

where the operator P is the Leray orthogonal projector, $P: H_0^1 \rightarrow V$ and $P: L^2 \rightarrow H$, respectively. Weak formulations of the NSE will use the bilinear Dirichlet form $a(\cdot, \cdot)$ and trilinear form $b(\cdot, \cdot, \cdot)$ as:

$$a(u, v) = (\nabla u, \nabla v)_H, \quad b(u, v, w) = (B(u, v), w)_H = (P((u \cdot \nabla)v), w)_H.$$

Again, thanks to the Poincaré inequality, the form $a(\cdot, \cdot)$ is actually an inner-product on V , and as noted above, the induced semi-norm $|\cdot|_{H^1(\Omega)} = a(\cdot, \cdot)^{1/2}$ is in fact a norm on V , equivalent to the H^1 -norm. *A priori* bounds and various symmetries can be derived for the trilinear form $b(\cdot, \cdot, \cdot)$; the results of this type that we will need are collected together in Appendix A.

A general weak formulation of the NSE (1.1)–(1.2) can be written as (cf. [32, 33, 4]):

Definition 2.1 (Weak Solutions of the NSE). *Given $f \in L^2([0, T]; V')$, a weak solution of the NSE satisfies $u \in L^2([0, T]; V) \cap C_w([0, T]; H)$, $du/dt \in L_{loc}^1((0, T]; V')$, and*

$$\left\langle \frac{du}{dt}, v \right\rangle + \nu a(u, v) + b(u, u, v) = \langle f, v \rangle, \quad \forall v \in V, \quad \text{for almost every } t, \quad (2.4)$$

$$u(0) = u_0. \quad (2.5)$$

Here, the space $C_w([0, T]; H)$ is the subspace of $L^\infty([0, T]; H)$ of weakly continuous functions, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and V' , where H is the Riesz-identified pivot space in the Gelfand triple $V \subset H = H' \subset V'$. Note that since the Stokes operator can be uniquely extended to $A: V \rightarrow V'$, and since it can be shown that $B: V \times V \rightarrow V'$ (cf. [4, 33] for both results), the functional form (2.3) still makes sense for weak solutions, and the total operator represents a mapping $V \rightarrow V'$.

Consider now two forcing functions $f, g \in L^2([0, \infty]; V')$ and corresponding weak solutions u and v to (2.3) in either the two- or three-dimensional case. Subtracting equations (2.3) for u and v yields an equation for the difference function $w = u - v$, namely

$$\frac{dw}{dt} + \nu Aw + B(u, u) - B(v, v) = f - g. \quad (2.6)$$

Since the residual of (2.6) lies in the dual space V' , for almost every t , we can consider duality pairing of (2.6) with a function in V , and in particular with $w \in V$, which yields

$$\left\langle \frac{dw}{dt}, w \right\rangle + \nu a(w, w) + b(u, u, w) - b(v, v, w) = \langle f - g, w \rangle \quad \text{for almost every } t.$$

Using the notation (2.1)–(2.2) going forward, it can be shown (cf. [32, 33]) that

$$\frac{1}{2} \frac{d}{dt} \|w\|_H^2 = \left\langle \frac{dw}{dt}, w \right\rangle \quad (2.7)$$

in the sense of distributions. Lemma A.4 in Appendix A establishes the symmetry relation $b(w, u, w) = b(u, u, w) - b(v, v, w)$, $\forall u, v, w \in V$, so the function $w = u - v$ must satisfy

$$\frac{1}{2} \frac{d}{dt} \|w\|_H^2 + \nu \|w\|_V^2 + b(w, u, w) = \langle f - g, w \rangle. \quad (2.8)$$

Equation (2.8) will be the starting point for our analysis of determining projections below. In the introduction, we highlighted an approximation property (1.3) that we will assume that a determining projection satisfies, and we will give an explicit example of such a projection below from [21, 23]. A useful consequence of property (1.3) that was noted in [23] is the following.

Lemma 2.2. *Let $R_N: V \rightarrow V_N \subset L^2(\Omega)$, $N = \dim(V_N) < \infty$, satisfy the following approximation inequality for some $\gamma > 0$:*

$$\|u - R_N u\|_{L^2(\Omega)} \leq C_1 N^{-\gamma} \|u\|_{H^1(\Omega)}. \quad (2.9)$$

Then the following inequalities hold:

$$\|u\|_{L^2(\Omega)} \leq 2C_1^2 N^{-2\gamma} \|u\|_{H^1(\Omega)} + 2\|R_N u\|_{L^2(\Omega)}^2, \quad (2.10)$$

$$\|u\|_{H^1(\Omega)} \geq [N^{2\gamma}/(2C_1^2)] \|u\|_{L^2(\Omega)} - [N^{2\gamma}/C_1^2] \|R_N u\|_{L^2(\Omega)}^2. \quad (2.11)$$

Proof. We start with squaring (2.9),

$$\|u - R_N u\|_{L^2(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 - 2(u, R_N u)_{L^2(\Omega)} + \|R_N u\|_{L^2(\Omega)}^2 \leq C_1^2 N^{-2\gamma} \|u\|_{H^1(\Omega)}^2.$$

Rearranging the inequality we have

$$\begin{aligned} \|u\|_{L^2(\Omega)}^2 &\leq C_1^2 N^{-2\gamma} \|u\|_{H^1(\Omega)}^2 - \|R_N u\|_{L^2(\Omega)}^2 + 2(u, R_N u)_{L^2(\Omega)} \\ &\leq C_1^2 N^{-2\gamma} \|u\|_{H^1(\Omega)}^2 - \|R_N u\|_{L^2(\Omega)}^2 + 2\|u\|_{L^2(\Omega)} \|R_N u\|_{L^2(\Omega)} \\ &= C_1^2 N^{-2\gamma} \|u\|_{H^1(\Omega)}^2 - \|R_N u\|_{L^2(\Omega)}^2 + \left(\frac{1}{\sqrt{2}} \|u\|_{L^2(\Omega)}\right) \left(2\sqrt{2} \|R_N u\|_{L^2(\Omega)}\right) \\ &\leq C_1^2 N^{-2\gamma} \|u\|_{H^1(\Omega)}^2 - \|R_N u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + 2\|R_N u\|_{L^2(\Omega)}^2, \end{aligned}$$

which after multiplying through by 2 and simplifying gives (2.10). Rearrangement of the terms in (2.10) then also gives (2.11) \square

Finally, we note key tools for establishing a number of *a priori* estimates in Sections 4 and 5 will be both classical and generalized Gronwall inequalities (cf. [9, 24]) which we include in Appendix A.

3. THE FRAMEWORK FOR CONSTRUCTING DETERMINING PROJECTIONS

We now give an overview of the general framework for constructing determining projections for the NSE for both two and three spatial dimensions, represented by Theorem 3.1 below. To make use of the framework to establish bounds on the number of determining degrees of freedom for weak solutions, one must assume, or establish, a single *a priori* bound for solutions to the equations, (inequality (3.2) below) and also provide a projection operator that satisfies a single approximation inequality (inequality (2.9) above).

Our earlier results in [21, 23] are included as particular instances of this framework, and we include in Appendix B the well-known *a priori* bounds for constant viscosity in the $d = 2$ and $d = 3$ cases that were used in [21, 23, 22]. The remainder of the article then turns to the necessary *a priori* bounds for non-constant viscosity.

Theorem 3.1 (Existence of Determining Projections for the NSE on domains $\Omega \subset \mathbb{R}^2$). *Let $f(t), g(t) \in V'$ be any two forcing functions satisfying*

$$\lim_{t \rightarrow \infty} \|f(t) - g(t)\|_{V'} = 0,$$

and let $u, v \in V$ be the corresponding weak solutions to (1.1)–(1.2) for $d = 2$. If there exists a projection operator $R_N: V \rightarrow V_N \subset L^2(\Omega)$, $N = \dim(V_N)$, satisfying

$$\lim_{t \rightarrow \infty} \|R_N(u(t) - v(t))\|_{L^2(\Omega)} = 0,$$

and satisfying for $\gamma > 0$ the approximation inequality

$$\|u - R_N u\|_{L^2(\Omega)} \leq C_1 N^{-\gamma} \|u\|_{H^1(\Omega)},$$

then

$$\lim_{t \rightarrow \infty} \|u(t) - v(t)\|_H = 0$$

holds if N is such that

$$\infty > N > C \left(\frac{1}{\nu^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'} \right)^{\frac{1}{\gamma}},$$

where C is a constant independent of ν and f .

Proof. Staying with the notation (2.1)–(2.2), we begin with equation (2.8), employing inequality (A.3) from Theorem A.3 in Appendix A, along with Cauchy-Schwarz and Young's inequalities, to yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_H^2 + \nu \|w\|_V^2 &\leq \|u\|_V \|w\|_H \|w\|_V + \|f - g\|_{V'} \|w\|_V \\ &\leq \frac{1}{\nu} \|u\|_V^2 \|w\|_H^2 + \frac{1}{\nu} \|f - g\|_{V'}^2 + \frac{\nu}{2} \|w\|_V^2. \end{aligned}$$

Equivalently, this is

$$\frac{d}{dt} \|w\|_H^2 + \nu \|w\|_V^2 - \frac{2}{\nu} \|u\|_V^2 \|w\|_H^2 \leq \frac{2}{\nu} \|f - g\|_{V'}^2.$$

To bound the second term on the left from below, we employ the approximation inequality (2.11) from Lemma 2.2, which yields

$$\frac{d}{dt} \|w\|_H^2 + \left(\frac{\nu N^{2\gamma}}{2C_1^2} - \frac{2}{\nu} \|u\|_V^2 \right) \|w\|_H^2 \leq \frac{2}{\nu} \|f - g\|_{V'}^2 + \frac{\nu N^{2\gamma}}{C_1^2} \|R_N w\|_{L^2(\Omega)}^2.$$

This is a differential inequality of the form

$$\frac{d}{dt} \|w\|_H^2 + \alpha \|w\|_H^2 \leq \beta,$$

with obvious definition of α and β .

The Generalized Gronwall Lemma A.6 can now be applied. Recall that we have assumed both $\|f - g\|_{V'} \rightarrow 0$ and $\|R_N w\|_{L^2(\Omega)} \rightarrow 0$ as $t \rightarrow \infty$. Since it is assumed that u and v , and hence w , are in V , so that all other terms appearing in α and β remain bounded, it must hold that

$$\lim_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau) d\tau = 0, \quad \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \alpha^-(\tau) d\tau < \infty.$$

It remains to verify that for some fixed $T > 0$,

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) d\tau > 0.$$

This means we must verify the following inequality for some fixed $T > 0$:

$$N^{2\gamma} > \frac{2C_1^2}{\nu} \left(\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \frac{2\|u\|_V^2}{\nu} d\tau \right) = \frac{4C_1^2}{\nu^2} \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u\|_V^2 d\tau. \quad (3.1)$$

The following *a priori* bound on any weak solution can be shown to hold (Lemma B.2 in Appendix B):

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_H^2 d\tau \leq \frac{2}{\nu^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2, \quad (3.2)$$

for any $T > c_\rho^2/\nu > 0$, where c_ρ is the best constant from the Poincaré inequality (Lemma A.2 in Appendix A). Therefore, if

$$N^{2\gamma} > 8C_1^2 \left(\frac{1}{\nu^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'} \right)^2 \geq \frac{4C_1^2}{\nu^2} \left(\frac{2}{\nu^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2 \right), \quad (3.3)$$

implying that (3.1) holds, then by the Gronwall Lemma A.6, it follows that

$$\lim_{t \rightarrow \infty} \|w(t)\|_H = \lim_{t \rightarrow \infty} \|u(t) - v(t)\|_H = 0.$$

□

Remark 3.2. *Theorem 3.1 for the $d = 2$ case can be extended to $d = 3$ in several ways, which we will not reproduce here. For example, in [23] a finite energy dissipation assumption was used to extend Theorem 3.1 to $d = 3$ case; a different approach for the $d = 3$ case is taken in [21].*

Remark 3.3. *In the case of constant viscosity, the required a priori estimate in (3.2) in the proof of Theorem 3.1 is provided by Lemma B.2 in Appendix B). For the case of time-varying viscosity, the required estimate is provided by Lemma C.1 or Lemma 4.2 in Section 4. For the case of our simple model of space-varying viscosity, the required estimate is provided by Proposition 5.3 in Section 5.*

4. A PRIORI ESTIMATES FOR TIME-VARYING VISCOSITY

In this section, we develop the *a priori* estimates needed to apply the determining projection framework from §3 (Theorem 3.1) to the case of a time-varying viscosity function $\nu = \nu(t)$. The first is an L^2 estimate that is used to prove the second and third estimates, following the strategy for the case of constant viscosity (see Appendix B). The second estimate is what is needed for use with Theorem 3.1 in different contexts. The time-varying viscosity is assumed to satisfy $\nu \in L^1(0, T)$ and obey the *a priori* pointwise bounds:

$$0 < \underline{\nu} \leq \nu(t) \leq \bar{\nu} < +\infty, \quad \forall t \in (0, T), \quad (4.1)$$

where

$$\underline{\nu} = \inf_{0 < t < T} \nu(t), \quad \bar{\nu} = \sup_{0 < t < T} \nu(t). \quad (4.2)$$

The first estimate gives a bound on the L^2 -norm of a weak solution to (1.1)–(1.2).

Lemma 4.1 (L^2 -Estimates, time-varying viscosity). *Let $\nu \in L^1(0, T)$ and assume that (4.1)–(4.2) hold. Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (1.1)–(1.2), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. It holds that*

$$\limsup_{t \rightarrow \infty} \|u(t)\|_H^2 \leq \frac{\bar{K}}{\underline{\nu}} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2, \quad (4.3)$$

where $K = \limsup_{t \rightarrow \infty} \int_0^t e^{-\phi_s(t)/c_\rho^2} ds$, $\phi_s(t) = \int_s^t \nu(z) dz$, and c_ρ is the Poincaré constant.

Proof. Beginning with equation (2.4) for $v = u$, using (2.7), and noting that Theorem A.4 in Appendix A ensures $b(u, u, u) = 0$, we are left with

$$\frac{1}{2} \frac{d}{dt} \|u\|_H^2 + \nu \|u\|_V^2 \leq \|f\|_{V'} \|u\|_V.$$

Applying Young's inequality leads to

$$\frac{d}{dt} \|u\|_H^2 + 2\nu \|u\|_V^2 \leq \left(\sqrt{\frac{2}{\nu}} \|f\|_{V'} \right) \left(\sqrt{2\nu} \|u\|_V \right) \leq \frac{1}{\nu} \|f\|_{V'}^2 + \nu \|u\|_V^2,$$

which gives then

$$\frac{d}{dt} \|u\|_H^2 + \nu \|u\|_V^2 \leq \frac{1}{\nu} \|f\|_{V'}^2. \quad (4.4)$$

Employing the Poincaré inequality we end up with

$$\frac{d}{dt} \|u\|_H^2 + \frac{\nu}{c_\rho^2} \|u\|_H^2 \leq \frac{1}{\nu} \|f\|_{V'}^2.$$

This is a differential inequality for $\|u(t)\|_H^2$, and by Gronwall's Inequality (Lemma A.5) it holds that

$$\begin{aligned} \|u(t)\|_H^2 &\leq \|u(r)\|_H^2 e^{-\int_r^t \nu(\tau)/c_\rho^2 d\tau} + \int_r^t \frac{1}{\nu(s)} \|f(s)\|_{V'}^2 e^{-\int_s^t \nu(\tau)/c_\rho^2 d\tau} ds \\ &= \|u(r)\|_H^2 e^{-1/c_\rho^2 \int_r^t \nu(\tau) d\tau} + \int_r^t \frac{1}{\nu} e^{-1/c_\rho^2 \int_s^t \nu(\tau) d\tau} \|f(s)\|_{V'}^2 ds \\ &= \|u(r)\|_H^2 e^{-\phi_r(t)/c_\rho^2} + \int_r^t \frac{1}{\nu} e^{-\phi_s(t)/c_\rho^2} \|f(s)\|_{V'}^2 ds \\ &\leq \|u(r)\|_H^2 e^{-\phi_r(t)/c_\rho^2} + \sup_{r \leq \delta \leq t} \frac{1}{\nu(\delta)} \|f(\delta)\|_{V'}^2 \int_r^t e^{-\phi_s(t)/c_\rho^2} ds \\ &\leq \|u(r)\|_H^2 e^{\phi_r(t)/c_\rho^2} + \frac{K}{\underline{\nu}} \sup_{r \leq \delta \leq t} \|f(\delta)\|_{V'}^2, \end{aligned}$$

where $\phi_c(t) = \int_c^t \nu(z) dz$ and $K = \int_0^t e^{-\phi_s(t)/c_\rho^2} ds$, which must hold for every $r \in (0, t]$. Taking the $\limsup_{t \rightarrow \infty}$ of both sides of the inequality leaves (4.3) with obvious definition of \bar{K} . \square

The following estimate gives the a bound on the time-averaged H^1 -semi-norm of a weak solution to (1.1)–(1.2).

Lemma 4.2 (Time-averaged H^1 -Estimates, time-varying viscosity). *Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (1.1)–(1.2), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. Then for every T with $T \geq c_\rho^2/\underline{\nu} > 0$ it holds that*

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{\bar{K} \underline{\nu}^3 + c_\rho^2}{\underline{\nu}^2 c_\rho^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2, \quad (4.5)$$

where $\bar{K} = \limsup_{t \rightarrow \infty} \int_0^t e^{-\phi_s(t)/c_\rho^2} ds$, $\phi_s(t) = \int_s^t \nu(z) dz$, and c_ρ is the Poincaré constant.

Proof. We begin with (4.4), which was

$$\frac{d}{dt} \|u\|_H^2 + \nu \|u\|_V^2 \leq \frac{1}{\nu} \|f\|_{V'}^2.$$

Dividing by $\nu(t)$ and integrating from t to $t + T$ with $T > 0$ gives

$$\int_t^{t+T} \frac{1}{\nu(\tau)} \frac{d}{d\tau} \|u(\tau)\|_H^2 d\tau + \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \int_t^{t+T} \frac{1}{\nu^2(\tau)} \|f(\tau)\|_{V'}^2 d\tau.$$

Integrating by parts the left-most term gives

$$\begin{aligned} & \frac{1}{\nu(t+T)} \|u(t+T)\|_H^2 - \frac{1}{\nu(t)} \|u(t)\|_H^2 + \int_t^{t+T} \frac{1}{\nu^2(\tau)} \|u(\tau)\|_H^2 d\tau \\ & + \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \int_t^{t+T} \frac{1}{\nu^2(\tau)} \|f(\tau)\|_V^2 d\tau \end{aligned}$$

Dropping the positive first and third terms on the left and bounding the integral on the right gives

$$\int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{1}{\nu(t)} \|u(t)\|_H^2 + T \sup_{t \leq s \leq t+T} \frac{1}{\nu^2(s)} \|f(s)\|_V^2.$$

Taking the $\limsup_{t \rightarrow \infty}$ of both sides, and dividing by T , gives

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau & \leq \frac{1}{T} \limsup_{t \rightarrow \infty} \frac{1}{\nu(t)} \|u(t)\|_H^2 + \limsup_{t \rightarrow \infty} \frac{1}{\nu^2(t)} \|f(t)\|_V^2 \\ & \leq \frac{1}{\underline{\nu} T} \limsup_{t \rightarrow \infty} \|u(t)\|_H^2 + \frac{1}{\underline{\nu}^2} \limsup_{t \rightarrow \infty} \|f(t)\|_V^2. \end{aligned}$$

Using the estimate from Lemma 4.1 and bounding the right-most term gives then

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \left(\frac{1}{\underline{\nu} T} \cdot \frac{K}{\underline{\nu}} + \frac{1}{\underline{\nu}^2} \right) \limsup_{t \rightarrow \infty} \|f(t)\|_V^2.$$

Since $T \geq c_\rho^2 / \underline{\nu} > 0$, we end up with (4.5). \square

Remark 4.3. If one takes $\nu(t) \equiv c$, we find that $K = c_\rho^2 / \underline{\nu}$, recovering the bounds for constant viscosity in both of the above estimates for time-varying viscosity (see Appendix B).

5. WEAK FORMULATION AND ESTIMATES FOR SPACE-VARYING VISCOSITY

The remaining two sections of the notes are first steps to understanding the case of space-varying viscosity, and are both *preliminary* and *somewhat speculative*. In this section, we attempt to develop a weak formulation that is appropriate for viscosity that can vary with the spatial location. In the next section, we establish some preliminary *a priori* bounds for the space-varying case. As was the case for the time-varying viscosity, we will make some basic assumptions on the now space-varying viscosity:

$$0 < \underline{\nu} \leq \nu(x) \leq \bar{\nu} < +\infty, \quad \forall x \in \Omega, \quad (5.1)$$

where

$$\underline{\nu} = \inf_{x \in \Omega} \nu(x), \quad \bar{\nu} = \sup_{x \in \Omega} \nu(x). \quad (5.2)$$

Let us consider the effects of a space-varying viscosity on equations (2.4) and (2.5). Our interest here is to develop a weak formulation analogous to (2.4), but in which the viscosity is allowed to be space-varying, with its gradient is not necessarily zero. Unlike with the time-dependent case, the NSE will now require an extra term $\nabla \nu \cdot \nabla u$ that we will call the *viscosity-velocity divergence term*. If $\nabla \nu \neq 0$, then we must consider νA to be the modified Stokes operator. We will assume that $\nu \in W$, where W is an appropriate Banach space that will be determined later, such that νA remains a bounded linear map, where A is the stokes operator as in the earlier discussion. The term $\nu a(u, \eta)$ appearing in (2.4) now becomes $a(\nu u, \eta)$. We note that $\nu a(u, \eta)$ is bounded from below by $\nu |u|_{H^1(\Omega)}^2$ and $a(\nu u, \eta)$ is bounded from below by $|\nu u|_{H^1(\Omega)}^2$.

We begin with the NSE without consideration of viscosities λ and ν as constants. This can be written as follows from [31]:

$$\rho \frac{Du}{Dt} = \rho f - \nabla p + \nabla(\lambda \nabla \cdot u) + \nabla \cdot (2\nu D)$$

where $D_{rs} = \frac{1}{2}(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r})$ is the symmetric component of the gradient of velocity, often referred to as the deformation tensor or rate-of-strain tensor, and $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \cdot \nabla u$ is the material derivative. The following hold:

$$\begin{aligned} \nabla \cdot (2\nu D) &= \nabla(\nu \nabla \cdot u) + \nabla \cdot (\nu \nabla u), \\ \nabla \cdot (\nu \nabla u) &= [\nu, u] + \nu \nabla^2 u, \end{aligned}$$

where $[\nu, u]_i = \nabla \nu \cdot \nabla u_i$ (we will use $\nabla \nu \cdot \nabla u$ to denote this term), and (assuming the fluid is incompressible)

$$\nabla \cdot u = 0.$$

With this, we can write

$$\rho \frac{Du}{Dt} = \rho f - \nabla p + \nu \nabla^2 u + \nabla \nu \cdot \nabla u \quad (5.3)$$

$$u(0) = u_0 \quad (5.4)$$

Multiplying both sides by a test function $\eta : \mathbb{R}^3 \rightarrow \mathbb{R}$, integrating over an appropriate domain $\Omega \subset \mathbb{R}^3$, we find that

$$\int_{\Omega} \rho \frac{\partial u}{\partial t} \cdot \eta - \int_{\Omega} \nu \nabla^2 u \cdot \eta + \int_{\Omega} \rho (u \cdot \nabla) u \cdot \eta + \int_{\Omega} \nabla p \cdot \eta - \int_{\Omega} (\nabla \nu \cdot \nabla u) \cdot \eta = \int_{\Omega} \rho f \cdot \eta$$

We reverse-integrate by parts the diffusive term $\nu \nabla^2 u \cdot \eta$ and pressure term $\nabla p \cdot \eta$, and use the Divergence Theorem:

$$\begin{aligned} \int_{\Omega} \nabla p \cdot \eta &= - \int_{\Omega} p \nabla \cdot \eta + \int_{\partial \Omega} p \eta \cdot \hat{n} \\ - \int_{\Omega} \nabla^2 u \cdot \nu \eta &= \int_{\Omega} \nabla u \cdot \nabla(\nu \eta) - \int_{\partial \Omega} \nu \frac{\partial u}{\partial \hat{n}} \cdot \eta \\ &= \int_{\Omega} \nu \nabla u \cdot \nabla \eta + \int_{\Omega} \nabla u \nabla \nu \eta - \int_{\partial \Omega} \frac{\partial u}{\partial \hat{n}} \cdot \nu \eta \end{aligned}$$

Substituting back, the result is

$$\begin{aligned} \int_{\Omega} \rho \frac{\partial u}{\partial t} \cdot \eta + \int_{\Omega} \nu \nabla u \cdot \nabla \eta + \int_{\Omega} \nabla u \cdot \nabla \nu \eta + \int_{\Omega} \rho (u \cdot \nabla) u \cdot \eta \\ - \int_{\Omega} p \nabla \cdot \eta - \int_{\Omega} (\nabla \nu \cdot \nabla u) \cdot \eta \\ = \int_{\Omega} \rho f \cdot \eta + \int_{\partial \Omega} (\nu \frac{\partial u}{\partial \hat{n}} - p \hat{n}) \cdot \eta \end{aligned}$$

Choosing the test function η so that $\eta = 0$ on $\partial \Omega$ removes the term involving the boundary integral. The divergence constraint $\nabla \cdot u = 0$ is now $\int_{\Omega} q \nabla \cdot u = 0 \forall q \in Q = L^2(\Omega)$. With this, we can write the weak formulation of the NSE:

Find $u \in U = L^2((0, T); V)$ such that

$$\begin{aligned} \int_{\Omega} \rho \frac{\partial u}{\partial t} \cdot \eta + \int_{\Omega} \nu \nabla u \cdot \nabla \eta + \int_{\Omega} \nabla u \cdot \nabla \nu \eta + \int_{\Omega} \rho (u \cdot \nabla) u \cdot \eta \\ - \int_{\Omega} p \nabla \cdot \eta - \int_{\Omega} (\nabla \nu \cdot \nabla u) \cdot \eta \\ = \int_{\Omega} \rho f \cdot \eta, \quad \forall \eta \in V = \mathbf{H}_0^1(\Omega), \\ \int_{\Omega} q \nabla \cdot u = 0, \quad \forall q \in Q = L^2(\Omega). \end{aligned}$$

Employing again the Leray orthogonal projector P to incorporate the divergence constraint into our functional framework, we have the final weak formulation that allows for variable viscosity: Given $f \in L^2((0, T); H)$, if $u \in L^2((0, T); V)$ satisfies

$$\left(\frac{du}{dt}, \eta \right) + a(\nu u, \eta) + b(u, u, \eta) - (a(\nu, u), \eta) = (f, \eta), \quad \forall \eta \in V, \quad (5.5)$$

$$u(0) = u_0 \quad (5.6)$$

then u will be called a weak solution of the NSE with space-varying viscosity.

To go further with this analysis, we will need some type of a bound from below for the rather inconvenient term $a(\nu u, u)$, involving something more useful, such as some multiple of $\|u\|_{H^1(\Omega)}$. Under suitable regularity assumptions on ν and u , we have the following.

Proposition 5.1. *Let $\nu \in H^1(\Omega)$ satisfy (5.1)–(5.2), and let $u \in V \cap H^2(\Omega)$. Then*

$$a(\nu u, u) \geq \underline{\nu} |u|_{H^1(\Omega)}^2.$$

Proof. We first use the definition of the bilinear form $a(\cdot, \cdot)$ to write:

$$\begin{aligned} a(\nu u, u) &= (\nabla[\nu u], \nabla u)_{L^2(\Omega)} = \int_{\Omega} \nabla[\nu u] \cdot \nabla u = \int_{\Omega} (\nabla \nu) u \nabla u + \int_{\Omega} \nu (\nabla u)^2 \\ &\geq \int_{\Omega} (\nabla \nu) u \nabla u + \underline{\nu} \int_{\Omega} (\nabla u)^2 \\ &= \int_{\Omega} (\nabla \nu) u \nabla u + \underline{\nu} |u|_{H^1(\Omega)}^2. \end{aligned}$$

We are done if we can show that $\int_{\Omega} (\nabla \nu) u \nabla u \geq 0$. To this end, we integrate by parts to find that

$$\begin{aligned} \int_{\Omega} (\nabla \nu) u \nabla u &= \int_{\partial \Omega} \nu u \nabla u - \int_{\Omega} \nu u \nabla^2 u - \int_{\Omega} \nu (\nabla u)^2 \\ &\geq \int_{\partial \Omega} \nu u \nabla u - \int_{\Omega} \nu u \nabla^2 u - \bar{\nu} |u|_{H^1(\Omega)}^2. \end{aligned}$$

Note that $\int_{\partial \Omega} \nu u \nabla u$ vanishes due to the compact support of functions in V . It remains to show that $\int_{\Omega} \nu u \nabla^2 u + \bar{\nu} |u|_{H^1(\Omega)}^2 \leq 0$; showing $\int_{\Omega} u \nabla^2 u + |u|_{H^1(\Omega)}^2 \leq 0$ will suffice.

Integrating by parts the integral term gives

$$\int_{\Omega} u \nabla^2 u = \int_{\partial \Omega} u \nabla u - \int_{\Omega} (\nabla u)^2 = -|u|_{H^1(\Omega)}^2.$$

Thus, $\int_{\Omega} u \nabla^2 u + |u|_{H^1(\Omega)}^2 \leq 0$, which is what we were after. \square

We now establish some basic preliminary *a priori* bounds for the weak formulation of the simplified space-varying viscosity NSE model.

The first estimate gives a bound on the L^2 -norm of a weak solution to (1.1)–(1.2).

Proposition 5.2 (L^2 -Estimates, space-varying viscosity). *Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (5.3)–(5.4), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, and assume that Proposition 5.1 holds. Then it holds that*

$$\limsup_{t \rightarrow \infty} \|u(t)\|_H^2 \leq \frac{c_\rho^2}{2\underline{\nu}^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2 + \frac{c_\rho^2}{2\underline{\nu}^2} \limsup_{t \rightarrow \infty} \|\nabla \nu \cdot \nabla u(t)\|_W^2, \quad (5.7)$$

where c_ρ is the constant from the Poincaré inequality (Lemma A.2 in Appendix A), and $\underline{\nu} = \inf_\Omega \nu > 0$.

Proof. Beginning with (5.5) for $\eta = u$, using (2.7), noting that Theorem A.4 guarantees $b(u, u, u) = 0$, and under the assumption that Proposition 5.1 holds, we can start with

$$\frac{1}{2} \frac{d}{dt} \|u\|_H^2 + \underline{\nu} |u|_V^2 \leq \|f\|_{V'} |u|_V + \|\nabla \nu \cdot \nabla u\|_W |u|_V.$$

Applying Young's Inequality leads to

$$\begin{aligned} \frac{d}{dt} \|u\|_H^2 + 2\underline{\nu} |u|_V^2 &\leq (\sqrt{\underline{\nu}} \|f\|_{V'}) \left(\frac{1}{\sqrt{\underline{\nu}}} |u|_V \right) + (\sqrt{\underline{\nu}} \|\nabla \nu \cdot \nabla u\|_W) \left(\frac{1}{\sqrt{\underline{\nu}}} |u|_V \right) \\ &\leq \frac{\underline{\nu}}{2} \|f\|_{V'}^2 + \frac{1}{2\underline{\nu}} |u|_V^2 + \frac{\underline{\nu}}{2} \|\nabla \nu \cdot \nabla u\|_W^2 + \frac{1}{2\underline{\nu}} |u|_V^2, \end{aligned}$$

which gives then

$$\frac{d}{dt} \|u\|_H^2 + \underline{\nu} |u|_V^2 \leq \frac{1}{2\underline{\nu}} \|f\|_{V'}^2 + \frac{1}{2\underline{\nu}} \|\nabla \nu \cdot \nabla u\|_W^2. \quad (5.8)$$

Employing the Poincaré inequality, we we end up with

$$\frac{d}{dt} \|u\|_H^2 + \frac{\underline{\nu}}{c_\rho^2} \|u\|_H^2 \leq \frac{1}{2\underline{\nu}} \|f\|_{V'}^2 + \frac{1}{2\underline{\nu}} \|\nabla \nu \cdot \nabla u\|_W^2.$$

This is a differential inequality for $\|u(t)\|_H^2$, and by Gronwall's Inequality (Lemma A.5) it holds that

$$\begin{aligned} \|u(t)\|_H^2 &\leq \|u(r)\|_H^2 e^{-\int_r^t \underline{\nu}/c_\rho^2 d\tau} + \int_r^t \frac{1}{2\underline{\nu}} (\|f(s)\|_{V'}^2 + \|\nabla \nu \cdot \nabla u(s)\|_W^2) e^{-\int_r^t \underline{\nu}/c_\rho^2 d\tau} ds \\ &= \|u(r)\|_H^2 e^{-\underline{\nu}(t-r)/c_\rho^2} + \int_r^t \frac{1}{2\underline{\nu}} e^{-\underline{\nu}(t-s)/c_\rho^2} (\|f(s)\|_{V'}^2 + \|\nabla \nu \cdot \nabla u(s)\|_W^2) ds \\ &\leq \|u(r)\|_H^2 e^{-\underline{\nu}(t-r)/c_\rho^2} \\ &\quad + \frac{1}{2\underline{\nu}} \sup_{r \leq \delta \leq t} (\|f(\delta)\|_{V'}^2 + \|\nabla \nu \cdot \nabla u(\delta)\|_W^2) \int_r^t e^{\underline{\nu}(t-s)/c_\rho^2} ds \\ &= \|u(r)\|_H^2 e^{-\underline{\nu}(t-r)/c_\rho^2} \\ &\quad + \frac{1}{2\underline{\nu}} \sup_{r \leq \delta \leq s} (\|f(\delta)\|_{V'}^2 + \|\nabla \nu \cdot \nabla u(\delta)\|_W^2) \frac{c_\rho}{\underline{\nu}^2} \left(e^0 - e^{-\underline{\nu}(t-r)/c_\rho^2} \right), \end{aligned}$$

or more simply

$$\|u(t)\|_H^2 \leq \|u(r)\|_H^2 e^{\underline{\nu}(t-r)/c_\rho^2} + \frac{c_\rho^2}{2\underline{\nu}^2} \sup_{r \leq \delta \leq t} (\|f(\delta)\|_{V'}^2 + \|\nabla \nu \cdot \nabla u(\delta)\|_W^2),$$

which must hold for every $r \in (0, t]$. Taking the $\limsup_{t \rightarrow \infty}$ of both sides of the inequality leaves (5.7) \square

The second estimate gives a bound on the time-averaged H^1 -semi-norm of a weak solution to (5.3)–(5.4).

Proposition 5.3 (Time-averaged H^1 -Estimates, space-varying viscosity). *Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (5.3)–(5.4), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, and assume that Proposition 5.1 holds. Then for every T with $T \geq c_\rho^2/\underline{\nu} > 0$ it holds that*

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} |u(\tau)|_V^2 d\tau \leq \frac{1}{\underline{\nu}^2} \left(\limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2 + \limsup_{t \rightarrow \infty} \|\nabla \nu \cdot \nabla u\|_W^2 \right), \quad (5.9)$$

where c_ρ is the constant from the Poincaré inequality (Lemma A.2 in Appendix A), and $\underline{\nu} = \inf_\Omega \nu > 0$.

Proof. We can begin with (5.8) from the proof of Proposition 5.2, which was

$$\frac{d}{dt} \|u\|_H^2 + \underline{\nu} |u|_V^2 \leq \frac{1}{2\underline{\nu}} \|f\|_{V'}^2 + \frac{1}{2\underline{\nu}} \|\nabla \nu \cdot \nabla u\|_W^2.$$

Integrating from t to $t+T$ with $T \geq 0$ gives

$$\|u(t+T)\|_H^2 - \|u(t)\|_H^2 + \underline{\nu} \int_t^{t+T} |u(\tau)|_V^2 d\tau \leq \frac{1}{2\underline{\nu}} \int_t^{t+T} \|f(\tau)\|_{V'}^2 + \frac{1}{2\underline{\nu}} \int_t^{t+T} \|\nabla \nu \cdot \nabla u\|_W^2$$

Dropping the positive first term on the left and bounding the integral on the right gives

$$\int_t^{t+T} |u(\tau)|_V^2 \leq \frac{1}{\underline{\nu}} \|u(t)\|_H^2 + \frac{T}{2\underline{\nu}^2} \sup_{t \leq s \leq t+T} \|f(s)\|_{V'}^2 + \frac{T}{2\underline{\nu}^2} \sup_{t \leq s \leq t+T} \|\nabla \nu \cdot \nabla u\|_W^2.$$

Taking the $\limsup_{t \rightarrow \infty}$ of both sides, and dividing by T , gives

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} |u(\tau)|_V^2 &\leq \frac{1}{\underline{\nu}T} \limsup_{t \rightarrow \infty} \|u(t)\|_H^2 + \frac{1}{2\underline{\nu}^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2 \\ &\quad + \frac{1}{2\underline{\nu}^2} \limsup_{t \rightarrow \infty} \|\nabla \nu \cdot \nabla u(t)\|_W^2 \end{aligned}$$

Using the estimate from Proposition 5.2 gives then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} |u(\tau)|_V^2 &\leq \left(\frac{1}{\underline{\nu}T} \cdot \frac{c_\rho^2}{2\underline{\nu}^2} + \frac{1}{2\underline{\nu}^2} \right) \\ &\quad \left(\limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2 + \limsup_{t \rightarrow \infty} \|\nabla \nu \cdot \nabla u(t)\|_W^2 \right) \end{aligned}$$

Since $T \geq c_\rho^2/\underline{\nu}$, we end up with (5.9) \square

Remark 5.4. *To use these estimates with the determining projection framework of §3 (Theorem 3.1), it remains to determine appropriate function spaces for ν so that terms involving ν and $\nabla \nu$ are well-defined and compatible with both the weak formulation, the theory for weak solutions u , and any estimates we established above for determining projections. Although ν spatially varies, it is taken here to be given as data, and one can reverse-engineer any assumptions needed for e.g. $\nabla \nu \cdot \nabla u$, or other terms involving ν , to be well-defined. Allowing for a more complicated class of variable viscosity, such as viscosity that varies with the velocity, would greatly complicate this discussion.*

APPENDIX A. SOME TECHNICAL TOOLS

Here is a collection of some standard technical tools that we use in the paper. Young's inequality is used repeatedly throughout.

Lemma A.1 (Young's Inequality). *For $a, b \geq 0$, $1 < p, q < \infty$, $1/p + 1/q = 1$, it holds that*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (\text{A.1})$$

Proof. See for example [27]. □

We use the Poincaré Inequality in several places; in our setting, it takes the following form for both the classical Sobolev space $H_0^1(\Omega)$ and the space of vector-valued functions $\mathbf{H}_0^1(\Omega)$.

Lemma A.2 (Poincaré Inequality). *If Ω is bounded, then it holds that*

$$\|u\|_{L^2(\Omega)} \leq c_\rho(\Omega) |u|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \quad (\text{A.2})$$

Proof. For example see [30]. □

In this paper, we use the notation H and V for $L^2(\Omega)$ and $H^1(\Omega)$, respectively. The following *a priori* bounds can be derived for the trilinear form $b(\cdot, \cdot, \cdot)$.

Lemma A.3 (Trilinear Form Bounds). *If $\Omega \subset \mathbb{R}^d$, then the trilinear form $b(u, v, w)$ is bounded on $V \times V \times V$ as follows, where $d = 2$ or $d = 3$ is the spatial dimension:*

$$d = 2 : \quad |b(u, v, w)| \leq 2^{1/2} \|u\|_{L^2(\Omega)}^{1/2} |u|_{H^1(\Omega)}^{1/2} |v|_{H^1(\Omega)} \|w\|_{L^2(\Omega)}^{1/2} |w|_{H^1(\Omega)}^{1/2}, \quad (\text{A.3})$$

$$d = 3 : \quad |b(u, v, w)| \leq 2 \|u\|_{L^2(\Omega)}^{1/4} |u|_{H^1(\Omega)}^{3/4} |v|_{H^1(\Omega)} \|w\|_{L^2(\Omega)}^{1/4} |w|_{H^1(\Omega)}^{3/4}. \quad (\text{A.4})$$

Moreover, from Hölder inequalities we have for $d = 2$ or $d = 3$:

$$|b(v, u, v)| \leq \|\nabla u\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)}^2. \quad (\text{A.5})$$

Proof. See [28, 32, 33, 4]. □

The following useful symmetries can be shown for the trilinear form.

Lemma A.4 (Trilinear Form Symmetries). *If $\Omega \subset \mathbb{R}^d$, then the trilinear form $b(u, v, w)$ on $V \times V \times V$ has the following symmetries:*

$$\begin{aligned} b(u, v, v) &= 0, \\ b(u, v, w) &= -b(u, w, v), \\ b(u - v, u, u - v) &= b(u, u, u - v) - b(v, v, u - v). \end{aligned}$$

Proof. See [32, 33, 4]. □

The classical Gronwall inequality is as follows.

Lemma A.5 (Gronwall Inequality). *If $\alpha(t)$ and $\beta(t)$ are real-valued and non-negative on $(0, \infty)$, and if the function $y(t)$ satisfies the following differential inequality:*

$$y'(t) + \alpha(t)y(t) \leq \beta(t), \quad \text{a.e. on } (0, \infty),$$

then $y(t)$ is bounded on $(0, \infty)$ by

$$y(t) \leq y(0)e^{-\int_0^t \alpha(\tau) d\tau} + \int_0^t \beta(s)e^{-\int_0^s \alpha(\tau) d\tau} ds.$$

Proof. See for example [27]. □

The following generalized Gronwall inequality is used repeatedly throughout Sections 4 and 5 to obtain *a priori* estimates.

Lemma A.6 (Generalized Gronwall Lemma). *Let $T > 0$ be fixed, and let $\alpha(t)$ and $\beta(t)$ be locally integrable and real-valued on $(0, \infty)$, satisfying:*

$$\liminf_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) d\tau = m > 0, \quad \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \alpha^-(\tau) d\tau = M < \infty,$$

$$\lim_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau) d\tau = 0,$$

where $\alpha^- = \max\{-\alpha, 0\}$ and $\beta^+ = \max\{\beta, 0\}$. If $y(t)$ is an absolutely continuous non-negative function on $(0, \infty)$, and $y(t)$ satisfies the following differential inequality:

$$y'(t) + \alpha(t)y(t) \leq \beta(t), \quad \text{a.e. on } (0, \infty),$$

then $\lim_{t \rightarrow \infty} y(t) = 0$.

Proof. See [9, 24]. □

APPENDIX B. A PRIORI ESTIMATES FOR CONSTANT VISCOSITY

Variations of the following two *a priori* bounds on solutions to the NSE can be found throughout the literature on the Navier-Stokes equations; cf. [32, 33, 4]. For example, Lemmas B.1 and B.2 below (both from [21]) are simple generalizations to $f \in V'$ of the bounds in e.g. [4], presented there for $f \in H$.

The first estimate gives a bound on the L^2 -norm of a weak solution to (1.1)–(1.2).

Lemma B.1 (L^2 -Estimates, constant viscosity). *Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (1.1)–(1.2), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. It holds that*

$$\limsup_{t \rightarrow \infty} \|u(t)\|_H^2 \leq \frac{c_p^2}{\nu^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2, \quad (\text{B.1})$$

where c_p is the constant from the Poincaré inequality.

Proof. Beginning with equation (2.4) for $v = u$, using (2.7), and noting that Theorem A.4 in Appendix A ensures $b(u, u, u) = 0$, we are left with

$$\frac{1}{2} \frac{d}{dt} \|u\|_H^2 + \nu \|u\|_V^2 \leq \|f\|_{V'} \|u\|_V.$$

Applying Young's inequality leads to

$$\frac{d}{dt} \|u\|_H^2 + 2\nu \|u\|_V^2 \leq \left(\sqrt{\frac{2}{\nu}} \|f\|_{V'} \right) \left(\sqrt{2\nu} \|u\|_V \right) \leq \frac{1}{\nu} \|f\|_{V'}^2 + \nu \|u\|_V^2,$$

which gives then

$$\frac{d}{dt} \|u\|_H^2 + \nu \|u\|_V^2 \leq \frac{1}{\nu} \|f\|_{V'}^2. \quad (\text{B.2})$$

Employing the Poincaré inequality we end up with

$$\frac{d}{dt} \|u\|_H^2 + \frac{\nu}{c_p^2} \|u\|_H^2 \leq \frac{1}{\nu} \|f\|_{V'}^2.$$

This is a differential inequality for $\|u(t)\|_H^2$, and by Gronwall's Inequality (Lemma A.5) it holds that

$$\begin{aligned} \|u(t)\|_H^2 &\leq \|u(r)\|_H^2 e^{-\int_r^t \nu/c_\rho^2 d\tau} + \int_r^t \frac{1}{\nu} \|f(s)\|_{V'}^2 e^{-\int_s^t \nu/c_\rho^2 d\tau} ds \\ &= \|u(r)\|_H^2 e^{-\nu(t-r)/c_\rho^2} + \int_r^t \frac{1}{\nu} e^{-\nu(t-s)/c_\rho^2} \|f(s)\|_{V'}^2 ds \\ &\leq \|u(r)\|_H^2 e^{-\nu(t-r)/c_\rho^2} + \frac{1}{\nu} \sup_{r \leq \delta \leq t} \|f(\delta)\|_{V'}^2 \int_r^t e^{-\nu(t-s)/c_\rho^2} ds \\ &= \|u(r)\|_H^2 e^{-\nu(t-r)/c_\rho^2} + \frac{1}{\nu} \sup_{r \leq \delta \leq t} \|f(\delta)\|_{V'}^2 \frac{c_\rho^2}{\nu} \left(e^0 - e^{-\nu(t-r)/c_\rho^2} \right), \end{aligned}$$

or more simply

$$\|u(t)\|_H^2 \leq \|u(r)\|_H^2 e^{-\nu(t-r)/c_\rho^2} + \frac{c_\rho^2}{\nu^2} \sup_{r \leq \delta \leq t} \|f(\delta)\|_{V'}^2,$$

which must hold for every $r \in (0, t]$. Taking the $\limsup_{t \rightarrow \infty}$ of both sides of the inequality leaves (B.1). \square

The second estimate gives a bound on the time-averaged H^1 -semi-norm of a weak solution to (1.1)–(1.2).

Lemma B.2 (Time-averaged H^1 -Estimates, constant viscosity). *Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (1.1)–(1.2), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. Then for every T with $T \geq c_\rho^2/\nu > 0$ it holds that*

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{2}{\nu^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2, \quad (\text{B.3})$$

where c_ρ is the constant from the Poincaré inequality.

Proof. We begin with (B.2), which was

$$\frac{d}{dt} \|u\|_H^2 + \nu \|u\|_V^2 \leq \frac{1}{\nu} \|f\|_{V'}^2.$$

Integrating from t to $t + T$ with $T > 0$ gives

$$\|u(t+T)\|_H^2 - \|u(t)\|_H^2 + \nu \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{1}{\nu} \int_t^{t+T} \|f(\tau)\|_{V'}^2 d\tau.$$

Dropping the positive first term on the left and bounding the integral on the right gives

$$\int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{1}{\nu} \|u(t)\|_H^2 + \frac{T}{\nu^2} \sup_{t \leq s \leq t+T} \|f(s)\|_{V'}^2.$$

Taking the $\limsup_{t \rightarrow \infty}$ of both sides, and dividing by T , gives

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{1}{\nu T} \limsup_{t \rightarrow \infty} \|u(t)\|_H^2 + \frac{1}{\nu^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2.$$

Using the estimate from Lemma B.1 gives then

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \left(\frac{c_\rho^2}{\nu^3 T} + \frac{1}{\nu^2} \right) \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2.$$

Since $T \geq c_\rho^2/\nu > 0$, we end up with (B.3). \square

APPENDIX C. ADDITIONAL A PRIORI ESTIMATES FOR TIME-VARYING VISCOSITY

The following estimate gives a bound on the time-averaged product of viscosity and the H^1 -semi-norm of the weak solution to (1.1)–(1.2).

Lemma C.1 (Time-averaged H^1 -Estimates, time-varying viscosity). *Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (1.1)–(1.2), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. Then for every T with $T \geq c_\rho^2/\underline{\nu} > 0$ it holds that*

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \nu(\tau) \|u(\tau)\|_V^2 d\tau \leq \frac{\bar{K}\underline{\nu} + c_\rho^2}{\underline{\nu}c_\rho^2} \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2, \quad (\text{C.1})$$

where $\bar{K} = \limsup_{t \rightarrow \infty} \int_0^t e^{-\phi_s(t)/c_\rho^2} ds$, $\phi_s(t) = \int_s^t \nu(z) dz$, and c_ρ is the Poincaré constant.

Proof. We begin with (4.4), which was

$$\frac{d}{dt} \|u\|_H^2 + \nu \|u\|_V^2 \leq \frac{1}{\nu} \|f\|_{V'}^2.$$

Integrating from t to $t + T$ with $T > 0$ gives

$$\|u(t+T)\|_H^2 - \|u(t)\|_H^2 + \int_t^{t+T} \nu(\tau) \|u(\tau)\|_V^2 d\tau \leq \int_t^{t+T} \frac{1}{\nu(\tau)} \|f(\tau)\|_{V'}^2 d\tau.$$

Dropping the positive first term on the left and bounding the integral on the right gives

$$\int_t^{t+T} \nu(\tau) \|u(\tau)\|_V^2 d\tau \leq \|u(t)\|_H^2 + T \sup_{t \leq s \leq t+T} \frac{1}{\nu(s)} \|f(s)\|_{V'}^2.$$

Taking the $\limsup_{t \rightarrow \infty}$ of both sides, and dividing by T , gives

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \nu(\tau) \|u(\tau)\|_V^2 d\tau \leq \frac{1}{T} \limsup_{t \rightarrow \infty} \|u(t)\|_H^2 + \limsup_{t \rightarrow \infty} \frac{1}{\nu(t)} \|f(t)\|_{V'}^2.$$

Using the estimate from Lemma 4.1 and bounding the right-most term gives then

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \nu(\tau) \|u(\tau)\|_V^2 d\tau \leq \left(\frac{K}{\underline{\nu}T} + \frac{1}{\underline{\nu}} \right) \limsup_{t \rightarrow \infty} \|f(t)\|_{V'}^2.$$

Since $T \geq c_\rho^2/\underline{\nu} > 0$, we end up with (C.1). \square

The following estimate is a variation of the other time-varying estimates from Lemmas 4.2 and C.1 and gives yet another slightly different bound on the time-averaged H^1 -semi-norm of a weak solution to 1.1–1.2.

Proposition C.2 (Time-averaged H^1 -Estimates, time-varying viscosity). *Let $u \in L^2((0, T); V)$ be a weak solution of the Navier-Stokes equations (1.1)–(1.2), with Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$. Then for every T with $T \geq c_\rho/\underline{\nu} > 0$ it holds that*

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \frac{1}{\nu(\tau)} |u(\tau)|_{H^1(\Omega)}^2 d\tau \leq C \limsup_{t \rightarrow \infty} \|f(t)\|_{L^2(\Omega)}^2$$

where C is dependent only on $\bar{K} = \limsup_{t \rightarrow \infty} \int_0^t e^{-\phi_s(t)/c_\rho^2} ds$, $\phi_s(t) = \int_s^t \nu(z) dz$, and c_ρ the Poincaré constant.

Proof. \square

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