DISCUSSION 8 SOLUTIONS

MATH 170A

Problem 0.1. Calculate the 2-norm of $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Solution. To do this we need to calculate the first singular value, which is the square root of the largest eigenvalue of $A^T A$. The matrix $A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, which has eigenvalue 2 (with multiplicity 2), since the eigenvalues of a diagonal matrix is its diagonal. Thus, $\|A\|_2 = \sqrt{\sigma_1} = \sqrt{2}$.

To calculate the SVD, clearly $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$. Since the eigenvectors of $A^T A$ can be $[1; 0]$ and $[0; 1]$, we have $V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. (I say “can be” since you could actually pick any orthonormal basis of $\mathbb{R}^2$.) Thus, since $A = U \Sigma V^T$, $U = AV \Sigma^{-1}$. (Recall that $V$ is orthogonal.) This means that $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. All together,

$$A = U \Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag*{□}$$

Problem 0.2. Explain why the basic power method $q, Aq, A^2q, \cdots$ (with scaling) converges to the eigenvector associated with the largest eigenvalue.

Solution. Because $A$ is semisimple, its eigenvectors $v_1, \cdots v_n$ form a basis for $\mathbb{R}^n$. Thus, for any initial vector $q$, we can rewrite $q$ as

$$q = c_1 v_1 + \cdots + c_n v_n,$$

for some constants $c_i$. Because $Av_i = \lambda_i v_i$, if we multiply by $A^j$, we get

$$A^j q = c_1 \lambda_1^j v_1 + \cdots + c_n \lambda_n^j v_n.$$

Next, we scale the whole problem by $\lambda_1^j$, to find

$$q_j := A^j q/\lambda_1^j = c_1 v_1 + c_2 v_2 \frac{\lambda_2^j}{\lambda_1^j} + \cdots + c_n v_n \frac{\lambda_n^j}{\lambda_1^j}.$$

(The symbol $:=$ means “is defined as equal to,” in case you haven’t seen that before.) Since $|\lambda_1| > |\lambda_j|$ (for $j > 1$), as $j \to \infty$, the fractions of eigenvalues go to zero. Thus, in the limit, the only remaining term is $c_1 v_1$. In other words
$q_j \rightarrow c_1 v_1$. Since $c_1 v_1$ is the dominant eigenvector of $A$, we have shown what we wanted to. □