Math 170A                               Name: \textit{Answer Key}
Final                                               Student ID number: 

Instructions: Answers without work may be given no credit at the grader’s discretion. The test is out of 63 points.

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}.
\]

1. Calculate the LU decomposition (using partial pivoting), of the matrix

\[
A = \begin{bmatrix}
-4 & -1 \\
3 & 2
\end{bmatrix}.
\]

Write out \( P, L \) and \( U \) as full matrices. (5 pts)

\[
\begin{bmatrix}
-4 & -1 \\
3 & 2
\end{bmatrix} \xrightarrow{\text{no pivot}} \begin{bmatrix}
-4 & -1 \\
\frac{-3}{4} & \frac{3}{4}
\end{bmatrix}
\]

So \quad \begin{align*}
P &= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \\
L &= \begin{bmatrix}
1 & 0 \\
-\frac{3}{4} & 1
\end{bmatrix} \\
U &= \begin{bmatrix}
-4 & -1 \\
0 & \frac{5}{4}
\end{bmatrix}
\end{align*}
2. List two advantages of implementing (the usually more complicated) code for sparse matrices, as compared to full matrices. (3 pts)

   1. Fewer flops. For example, if the semibandwidth is $s$, sparse backwards substitution takes $O(n \cdot s)$ flops versus $O(n^2)$ flops.

   2. Less memory usage, since don’t have to store the zeros.

   (students don’t have to explain)

3. List the four properties that a matrix norm satisfies. (1 pt each)

   1. $\|A\| > 0$ unless $A = 0$

   2. $\|cA\| = |c|\|A\|$ for any $c \in \mathbb{R}$

   3. $\|A + B\| \leq \|A\| + \|B\|$

   4. $\|AB\| \leq \|A\| \|B\|$.
4. Let \( u \in \mathbb{R}^n \) be a unit vector. Define \( Q = I - 2uu^T \).

(a) Explain what the operator \( Q \) does when it acts on a vector \( x \). A one word answer is not enough.

(2 pts)

\[ Q \text{ reflects } x \text{ over the hyperplane orthogonal to } u \text{ that goes through the origin.} \]

(b) Prove that \( Q \) is orthogonal. (3 pts)

Need to show \( Q^T = Q^{-1} \).

First: \( Q^T = Q \) since \( (I - 2uu^T)^T = (I - 2uu^T) = (I - 2uu^T) = I - 2uu^T \).

\[
Q^T Q = QQ^T = (I - 2uu^T)(I - 2uu^T) = I - 4uu^T + 4uu^T uu^T \\
\overset{\text{}}{\Rightarrow} \|u\|_2^2 = 1 \\
= I - 4uu^T + 4uu^T = I \\
\]

so \( Q^T = Q^{-1} = Q \).

5. Why are iterative methods necessary to calculate eigenvalues? (Hint: Consider the characteristic equation.) (3 pts)

The characteristic equation \( \det(A - \lambda I) = 0 \) is a polynomial equation of the same degree as the size of \( A \). Polynomial equations of degree greater than 4 cannot, in general, be solved exactly. Thus iterative (approximate) methods are necessary.
6. Calculate the QR decomposition of the matrix

\[ A = \begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix} \]

Write out \( Q \) and \( R \) as full matrices. (5 pts)

\[ U = \begin{bmatrix} 1 \\ \frac{3}{5} - \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{5} \end{bmatrix} \]

\[ \gamma = \frac{-9}{-5} = \frac{9}{5} \]

\[ \mathcal{C} = -5 \]

\[ (I - \gamma u u^T) \begin{bmatrix} -1 \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ z \end{bmatrix} - \frac{9}{5} \begin{bmatrix} 1 \\ -\frac{1}{5} \end{bmatrix} (-1 - \frac{2}{3}) \]

\[ = \begin{bmatrix} -1 \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} 3 \]

\[ = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

So \( Q = I - \gamma u u^T = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \)

\( R = \begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix} \)
7. Suppose I have three data points, \((t, y) = (-1, 5), (0, -1), (1, -3)\). Set up the least squares problem for this data, for the model that uses the basis functions \(\phi_1(t) = t\) and \(\phi_2(t) = t^2 - 1\). Then use the normal equations to solve this least squares problem. Finally, write out the model (using the given basis) that best fits the data. Note that there are three parts to this question. (8 pts)

\[
\begin{bmatrix}
1 & t & t^2 - 1 \\
0 & 0 & -1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
-1 \\
-3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 1 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 1 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
-8 \\
-1 \\
1 \\
\end{bmatrix}
\]

**Part 2**  
So normal eqns are 

\[
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
-8 \\
1 \\
\end{bmatrix}
\]

\[x_1 = -4, \quad x_2 = 1\]

**Part 3**  
So our model is \[y(t) = -4t + t^2 - 1\]
8. If I claim that an algorithm is backwards stable, what does that mean? Also, does this mean a solution calculated with this algorithm is accurate? Why or why not? (5 pts)

Backwards stable means that the approximate calculation with exact data is equivalent to an exact calculation with approximate data. The calculated soln may not be accurate if the condition number is large, since then an exact soln with approximate data need not be accurate.

Extra: Backwards stable means, as an example \( \hat{f}(A\|b) = (A+\delta A)\backslash(\delta b + b) \).

But we know

\[
\frac{\|\hat{x}\|}{\|x\|} \leq K(A) \left( \frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} + \frac{\|S\|}{\|A\|} \frac{\|\delta b\|}{\|b\|} \right)
\]

So the error is small if \( K(A) \) is also.

9. Suppose that I am solving a problem \( Ax = b \), where \( A \) is known to be exactly

\[
A = \begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix}.
\]

Suppose my data \( b \) is accurate to within 0.1%. To within what percentage must my solution \( x \) be accurate? (Use the 1-norm.) (5 pts)

\[
A^{-1} = \frac{1}{-8+3} \begin{bmatrix} 2 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 \\ 3/5 & 4/5 \end{bmatrix}
\]

\[\|A\|_1 = 7, \quad \|A^{-1}\|_1 = 1 \quad \Rightarrow \quad K_1(A) = 7 \cdot 1 = 7\]

\[
\frac{\|\hat{x}\|}{\|x\|} \leq K_1(A) \frac{\|\delta x\|}{\|\delta b\|},
\]

\[
= 7 \cdot 0.1\% = 0.7\%.
\]

So my solution is accurate to within 0.7% (This is ignoring rounding errors, which would be insignificant in my case.)
10. Does the Jacobi method of solving $Ax = b$ converge for all initial guesses if

$$A = \begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix}$$

Why or why not? (5 pts)

$$M = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$$

$$G = M^{-1} N = \begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{3}{2} & 0 \end{bmatrix}$$

$$\text{det}(G - \lambda I) = \lambda^2 - \frac{3}{8} = 0$$

$$\lambda = \pm \sqrt{\frac{3}{8}}$$

So $\rho(G) = \sqrt{\frac{3}{8}} < 1$, so it converges for all initial guesses.
11. Write a function for Matlab that takes as input a square matrix $A$, initial guess $x$, right side vector $b$, and a maximum number of iterations $maxiter$, then returns the solution $x$ of $Ax = b$, found using the Gauss-Seidel method. The program should run until it performs $maxiter$ iterations. Use only basic programming. (5 pts)

```matlab
function x = G-S(A, x, b, maxiter)

n = length(A);
for k = 1:maxiter
    oldx = x;
    x = b;
    for i = 1:n
        for j = 1:i-1
            x(i) = x(i) - A(i,j) * x(j);
        end
        for j = i+1:n
            x(i) = x(i) - A(i,j) * oldx(j);
        end
        x(i) = x(i)/A(i,i);
    end
end

(don't need to check nonsingularity or if it has converged.)
```
12. Apply three iterations of the power method with the matrix
\[ A = \begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix} \] with the starting guess \( x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

Write the estimate you get for the leading eigenvalue and its associated eigenvector. (5 pts)

\[
\begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

\[
\begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix}
\]

\[
\begin{bmatrix} -4 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{11}{3} \\ \frac{7}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -\frac{7}{11} \end{bmatrix}
\]

\[ \lambda \approx -\frac{11}{3} \quad V_1 \approx \begin{bmatrix} 1 \\ -\frac{7}{11} \end{bmatrix} \]

I don't care how \( V_1 \) is scaled.
13. Is the matrix

\[ A = \begin{bmatrix}
-4 & -1 \\
3 & 2
\end{bmatrix} \]

positive definite? Why or why not? (2 pts)

It is not. However, since it is not symmetric, we cannot use the Cholesky algorithm to check. Instead, we need to find \( \mathbf{x} \) so that \( \mathbf{x}^T A \mathbf{x} \leq 0 \).

\[ \begin{bmatrix} 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 \\
0 \end{bmatrix} = -4 < 0, \text{ so it's not pos. defin.} \]

14. Suppose we want to solve the Poisson equation on this grid:

![Grid diagram]

The outer points are boundary points, where boundary values for the solution will be given. Assuming the grid points are equally spaced, and assuming we order the points in the normal way, what will the semi-bandwidth of the resulting matrix \( A \) be, in order to estimate the solution of the Poisson equation using \( Ax = b \)? (3 pts)

There are 10 x 10 interior points, so the semi-bandwidth will be 10.

(extra: This is since the equation centered at one grid point only uses the info from the adjacent points.)