MATH 170A REVIEW QUESTIONS

(1) Is the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ positive definite? If so, find the Cholesky decomposition of the matrix.

(2) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Define a function $\|\cdot\|_A : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, by $\|x\|_A := \|Ax\|_2$. Is this function a norm? Why or why not?

(3) Let $A = \begin{pmatrix} -3 & 5 \\ 2 & 4 \end{pmatrix}$. Find a vector $x \in \mathbb{R}^2$ that lies in the direction of maximum magnification of $A$ with respect to the $\infty$-norm.

(4) Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{pmatrix}$. Find the LU decomposition of $A$.

(5) Let $A \in \mathbb{R}^{m \times n}$ where $m > n$. Given $b \notin \mathcal{R}(A)$, what conditions on $A$ will guarantee uniqueness of the least squares solution of $Ax = b$?

(6) Consider the ODE $u''(x) - u'(x) + u(x) = 2x$ for $x \in [0, 2]$ with $u(0) = 0$, $u(2) = 0$. Use the finite difference method to set up the equation $Au = b$. Use subintervals of length $1/3$.

(7) List the four properties that a matrix norm satisfies.

(8) Use Gaussian elimination with partial pivoting to solve the following system:

\[
\begin{align*}
2x_1 + x_2 + x_3 &= 5 \\
4x_1 - 6x_2 &= -2 \\
-2x_1 + 7x_2 + 2x_3 &= 9.
\end{align*}
\]

(9) Let $x + \delta x$ be an approximate solution to $Ax = b$. Define the residual $r = b - A(x + \delta x)$. Show $(A + \delta A)(x + \delta x) = b$ is satisfied with

\[
\delta A = \|x + \delta x\|_2^{-2}r(x + \delta x)^T.
\]

(10) Explain how QR factorization can be used to solve a least squares problem.

(11) Write MATLAB code that computes the LU decomposition of a given matrix $A$ without pivoting.

(12) Let $u \in \mathbb{R}^n$ be a unit vector. Define $Q = I - 2uu^T$, and show the following:

(a) $Q$ is symmetric.
(b) $Q^{-1} = Q$.
(c) $Q$ is orthogonal

(13) Suppose the matrix norm is induced from the vector norm.

(a) Prove that $\|Ax\| \leq \|A\| \cdot \|x\|$. 


(b) prove $||Ax|| \geq ||Ax^{-1}||$.
   \textit{Hint:} use (a) part.

(14) Suppose we want to calculate $b = Ax$, given $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$. Prove the following relationship between a perturbation in the input $x$ and the resulting perturbation in the output $b$:
   $$\frac{||\delta b||}{||b||} \leq \kappa(A) \frac{||\delta x||}{||x||}.$$
   \textit{Hint:} Use problem (13).

(15) Write MATLAB code that computes the 2-norm of a vector. Avoid the possible underflow or overflow as much as possible.
   \textit{Hint:} first properly scale the vector.

(16) State the singular value decomposition theorem for a real square matrix. Based on this theorem, give a formula for the 2-norm of a matrix.

(17) Consider the power method applied to the matrix $A = \begin{pmatrix} -6 & 1 \\ 3 & 2 \end{pmatrix}$. Will this method converge? If so, to which eigenvalue of $A$ will it converge?

(18) Write the MATLAB code that produces the solution of the system $Ax = b$, by solving only system involving $L$ and $U$, where $A = LU$ and $L$ upper triangular and $U$ lower triangular. What’s the complexity of this algorithm?

(19) Show that the iteration $x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b$ converges to the true solution of $Ax = b$ for every initial guess $x$ if and only if the spectral radius $\rho(G)$ of the iteration matrix $G = I - M^{-1}A$ is less than one. The average converge ratio never exceeds $\rho(G)$.

(20) Consider the overdetermined system.
   $$\begin{pmatrix} 1 \\ 2 \end{pmatrix}(x) = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$
   Calculate a $QR$ decomposition of the coefficient matrix, where $R$ is a 2x2 rotator, and $R$ is a 2x1 matrix. use the $QR$ decomposition to calculate the least squares solution.