DISCUSSION 2 SOLUTIONS

MATH 170A

**Problem 0.1.** Calculate \( L, U \) and \( P \) for \( A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \) using Gaussian elimination with partial pivoting.

**Solution.** In the following calculation, I will be overwriting \( A \) with my calculation. The fourth column is the permutation vector, not a vector \( b \). The multipliers will be in parentheses.

\[
\begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 4 & 3 \\ (0) & 1 & 3 & 1 \\ (1/2) & 1/2 & 1 & 2 \end{bmatrix}
\]

Thus,

\[
L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -1/2 \end{bmatrix}, \quad P = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

**Problem 0.2.** Watkins 2.2.24a,b: Let \( A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix} \).

(a) Calculate \( A^{-1} \) and \( \kappa_\infty(A) \).

(b) Find \( b, \delta b, x, \) and \( \delta x \) such that \( Ax = b, A(x + \delta x) = b + \delta b, \|\delta b\|_\infty/\|b\|_\infty \) is small, and \( \|\delta x\|_\infty/\|x\|_\infty \) is large.

**Solution.** (a) \( A^{-1} = \begin{bmatrix} 375 & -187 \\ -376 & 187.5 \end{bmatrix} \). The condition number is

\[
\kappa_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty = 1502 \cdot 563.5 = 846377.
\]

(b) In order for \( \|\delta b\|_\infty/\|b\|_\infty \) to be small and \( \|\delta x\|_\infty/\|x\|_\infty \) to be large, and since \( Ax = b \), we want \( x \) to point in (approximately) the direction of maximum magnification of \( A \), so that \( b \) is large compared to \( x \). Similarly, since \( A\delta x = \delta b \),
we want $\delta x$ to point in (approximately) the direction of minimum magnification of $A$, so that $\delta b$ is small compared to $\delta x$.

If we pick $x = [1; 1]$, $Ax = b = [749; 1502]$, and so $\|x\|_\infty = 1$ and $\|b\|_\infty = 1502$. Since $\|A\|_\infty = 1502$, we know this is the direction of maximum magnification.

The direction of minimum magnification is a bit harder. We know that min magnification is one over the max magnification of $A^{-1}$, so $1/563.5$ is our goal. We could guess that $\delta x = [1; -1]$ would be about ideal, and that’s not a bad guess. The “best” direction, though, can be found by picking $\delta b$ is the direction of maximum magnification of $A^{-1}$, i.e., $\delta b = [-1; 1]$. Then $A^{-1}\delta b = \delta x = [-562; 563.5]$. We could leave these like this, but let’s normalize these, and pick $\delta x = [-562/563.5; 1]$ and $\delta b = [-1/563.5, 1/563.5]$.

Finally, we measure. $\|\delta b\|_\infty/\|b\|_\infty = 1/563.5/1502 \approx 1.18 \cdot 10^{-6}$, and $\|\delta x\|_\infty/\|x\|_\infty = 1/1 = 1$. In other words, our error is about 100%, even though our $b$ was accurate to about 0.0001%.

**Problem 0.3.**
In Gaussian elimination with partial pivoting, the entries of $L$ all have $|l_{ij}| \leq 1$ (i.e., are all less than 1.) Explain why.

**Solution.** In Gaussian elimination with partial pivoting, the rows are exchanged so that the pivot is the largest available element of the column. The multipliers are then calculated as the other elements divided by the pivot. That means the multipliers must be less than or equal to one in absolute value. Since the $l_{ij}$ are either 0, 1, or a multiplier, that means $|l_{ij}| \leq 1$.

**Problem 0.4.** Explain how to use the residual in order to prove that a computed (approximate) solution to $Ax = b$ is accurate, and why this works.

**Solution.** Let’s say your computed solution is $\hat{x}$. Then $A\hat{x} \approx b$ (hopefully). More precisely, if we set $\delta x = \hat{x} - x$ and the residual $\hat{r} = b - A\hat{x}$, we can say $A(x + \delta x) = b - \hat{r}$. This is precisely in the form needed to use our estimate for when $b$ is perturbed, and we want to estimate the error on $x$. This estimate is

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}.$$

Plugging in our problem, we get

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq \kappa(A) \frac{\|\hat{r}\|}{\|b\|}.$$

In other words, if our condition number is small and our residual is small compared to $b$, then we know our solution must be accurate.