Math 170A
Midterm 2

Name: Answer Key
Student ID number: 

Instructions: Answers without work may be given no credit at the grader's discretion. The test is out of 32 points.

This cover page may be used at scratch paper. However, all final work must be on the page with the related question. Do not remove this sheet.
1. Solve this simple least squares problem: (4 pts)

\[
\begin{bmatrix}
-4 \\
3
\end{bmatrix}
\begin{bmatrix}
x \\
3
\end{bmatrix} =
\begin{bmatrix}
6 \\
3
\end{bmatrix}
\]

\[
\begin{bmatrix}
-4 \\
3
\end{bmatrix}
\begin{bmatrix}
3 \\
3
\end{bmatrix} =
\begin{bmatrix}
25 \\
3
\end{bmatrix}
\]

\[
\begin{bmatrix}
25 \\
3
\end{bmatrix}
\begin{bmatrix}
x \\
3
\end{bmatrix} =
\begin{bmatrix}
-15 \\
3
\end{bmatrix}
\]

\[
x = \frac{-15}{25} = -\frac{3}{5}
\]

2. What is the unit roundoff? (2 pts)

The unit roundoff is the largest possible relative error from a single floating point operation, in whatever floating point system we're using.
3. Briefly explain the difference between a priori and a posteriori estimates, and why each is important. (4 pts)

   a priori estimates are done before you do a calculation, and are thus an important theoretical tool to help you know accuracy before you do a calculation.

   A posteriori estimates are done after a calculation, usually using the calculated solution. They are important because they simple, and often give a better estimate than an a priori one, and are easier to find and prove than a priori estimates.

   (you didn't need to mention all that.)

4. Briefly explain what it means for an algorithm or a calculation to be backwards stable. (3 pts)

   Backwards stable means that an approximate solution or result from that calculation, using exact data, exactly solves (or is an exact result from) an approximate, but "very close" calculation.

   For instance, If \( \hat{x} \) approximately solves \( Ax = b \), our calculation was backwards stable if \( \hat{x} \) exactly solves \( A\hat{x} = b + \delta b \), if \( \|\delta b\| \ll \|b\| \).
5. In proving all sorts of estimates, the key inequality we used was $\|Ax\| \leq \|A\|\|x\|$. Prove this inequality is true for any induced matrix norm. (5 pts)

For any induced norm, $\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. Since $\|A\|$ is the largest possible, for any particular vector $\hat{x}$, $\|A\| \geq \frac{\|A\hat{x}\|}{\|\hat{x}\|}$

and so $\|A\hat{x}\| \leq \|A\| \|\hat{x}\|$. 

(I didn't expect you to say this, which is technically necessary:

That proof works if $\hat{x} \neq 0$. If $\hat{x} = 0$, though, it's obvious that $\|A \cdot 0\| \leq \|A\| \|0\| = 0$ since both sides are 0.

6. What is catastrophic cancelation? As part of your explanation, include an explicit example. (5 pts)

Catastrophic cancelation occurs when two already slightly wrong numbers that are very close are added together and almost perfectly cancel. This leads to a large relative error.

For instance, if $x = 1.234$, $y = 1.238$, in 3-digit arithmetic, $x$ is stored as $\hat{x} = 1.23$ and $y$ as $\hat{y} = 1.24$.

$\hat{y} - \hat{x} = .01$

but $y - x = .004$, so the absolute error is $0.006$, for a relative error of $\frac{0.006}{0.004} = 1.5$, so 150\%
7. Using some method, I found the (approximate) solution \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) to the matrix problem \( \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} x = \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} \). Without finding the true solution, bound (i.e., estimate) the relative error of my approximate solution. (If you calculate the true solution, you will receive 0 points. You can use whichever norm you feel is easiest/appropriate, but make sure to say which you are using.) (5 pts)

\[
\text{residual} = b - A\hat{x} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}
\]

\[
\frac{\|\delta x\|_{\infty}}{\|x\|_{\infty}} \leq K_{\infty}(A) \frac{\|\hat{x} - x\|_{\infty}}{\|b\|_{\infty}} \quad (I'm \text{ using } \infty \text{-norm}).
\]

\[
A^{-1} = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}, \quad \|A\|_{\infty} = 11, \quad \|A^{-1}\|_{\infty} = 9
\]

\[
K_{\infty}(A) = 99
\]

\[
\|A\|_{\infty} = 11
\]

\[
\|b\|_{\infty} = 0.9 , \quad \text{so}
\]

\[
\frac{\|\delta x\|_{\infty}}{\|x\|_{\infty}} \leq 99 \cdot \frac{1}{0.9} = 11
\]

so, pretty bad: My solution could be up to 1100% Wrong.

(The true answer was \( \begin{bmatrix} 1.3 \\ 2.5 \end{bmatrix} \), so actually somewhat accurate.)
8. Earlier, we mentioned that you should use partial pivoting with the LU decomposition because you have to avoid zeros on the diagonal. However, partial pivoting is also important in proving that the LU decomposition is (usually) backwards stable. Explain why using partial pivoting (as compared to no pivoting) helps make the LU decomposition more backwards stable. (Hint: How does partial pivoting affect $L$ and why does that help?) (4 pts)

Partial pivoting guarantees that all the entries in $I$ are $\leq 1$, and so $\|I\|_{\infty} \leq n$. Since

$$\frac{\|SA\|}{\|A\|} \leq 6n u \frac{\|L\| \|U\|}{\|A\|} + O(u^2),$$

(you didn't need that all)

having $\|L\|$ small helps the algorithm be backwards stable.

Without pivoting, even if $L$ exists, the norm $\|L\|$ could be large.