Problem 0.1. If I try to solve $Ax = b$ with $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$ using the Gauss-Seidel method, will it converge? If it does converge, by approximately what factor is the error reduced each iteration?

Solution. It will converge. For Gauss-Seidel, we have

$$G = M^{-1}N = (D - E)^{-1}F = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3/4 \end{bmatrix}.$$ 

It is easy to calculate that the eigenvalues of $G$ are 0 and $3/4$, and so $\rho(G) = 3/4 < 1$, and so it will converge. For each iteration, the error will be approximately multiplied by $3/4$. □

Problem 0.2. Consider a generic descent method for solving $Ax = b$. What is the function $J(y)$, and how is it used in this method? Describe in words and pictures how descent methods work in general.

Solution. The function $J(y) = \frac{1}{2}y^T Ay - y^T b$ is a quadratic quantity, often representing some kind of “energy” in a physical system. The quantity is set up so that the minimizer of $J(y)$ is exactly the solution to $Ax = b$. Thus, descent methods seek to minimize $J(y)$ via an iterative process. The general process is that you choose a direction (such as the gradient of $J$ direction for the steepest descent method), and a distance to go in that direction. Often, you will choose to go the distance such that you minimize $J$, at least in that direction. In the earlier figures, you can see some examples, where the directions are chosen in different ways. □