Problem 0.1. Watkins 2.2.24a,b: Let \( A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix} \).

(a) Calculate \( A^{-1} \) and \( \kappa_\infty(A) \).
(b) Find \( b, \delta b, x, \) and \( \delta x \) such that \( Ax = b, A(x + \delta x) = b + \delta b, \) \( \|\delta b\|_\infty / \|b\|_\infty \) is small, and \( \|\delta x\|_\infty / \|x\|_\infty \) is large.

Solution. (a) \( A^{-1} = \begin{bmatrix} 375 & -187 \\ -376 & 187.5 \end{bmatrix} \). The condition number is
\[
\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 1502 \cdot 563.5 = 846377.
\]
(b) In order for \( \|\delta b\|_\infty / \|b\|_\infty \) to be small and \( \|\delta x\|_\infty / \|x\|_\infty \) to be large, and since \( Ax = b \), we want \( x \) to point in (approximately) the direction of maximum magnification of \( A \), so that \( b \) is large compared to \( x \). Similarly, since \( A\delta x = \delta b \), we want \( \delta x \) to point in (approximately) the direction of minimum magnification of \( A \), so that \( \delta b \) is small compared to \( \delta x \).

If we pick \( x = [1; 1] \), \( Ax = b = [749; 1502] \), and so \( \|x\|_\infty = 1 \) and \( \|b\|_\infty = 1502 \). Since \( \|A\|_\infty = 1502 \), we know this is the direction of maximum magnification.

The direction of minimum magnification is a bit harder. We know that min magnification is one over the max magnification of \( A^{-1} \), so \( 1/563.5 \) is our goal. We could guess that \( \delta x = [1; -1] \) would be about ideal, and that’s not a bad guess. The “best” direction, though, can be found by picking \( \delta b \) is the direction of maximum magnification of \( A^{-1} \), i.e., \( \delta b = [-1; 1] \). Then \( A^{-1}\delta b = \delta x = [-562; 563.5] \). We could leave these like this, but let’s normalize these, and pick \( \delta x = [-562/563.5; 1] \) and \( \delta b = [-1/563.5, 1/563.5] \).

Finally, we measure. \( \|\delta b\|_\infty / \|b\|_\infty = 1/1502 \approx 1.18 \cdot 10^{-6} \), and \( \|\delta x\|_\infty / \|x\|_\infty = 1/1 = 1 \). In other words, our error is about 100%, even though our \( b \) was accurate to about 0.0001%.

Problem 0.2. Explain how to use the residual in order to prove that a computed (approximate) solution to \( Ax = b \) is accurate, and why this works.

Solution. Let’s say your computed solution is \( \hat{x} \). Then \( A\hat{x} \approx b \) (hopefully). More precisely, if we set \( \delta x = \hat{x} - x \) and the residual \( \hat{r} = b - A\hat{x} \), we can say \( A(x + \delta x) = b - \hat{r} \). This is precisely in the form needed to use our estimate for when \( b \) is perturbed, and we want to estimate the error on \( x \). This estimate is
\[
\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}.
\]
Plugging in our problem, we get
\[
\frac{\|\hat{x} - x\|}{\|x\|} \leq \kappa(A) \frac{\|\hat{r}\|}{\|b\|}
\]
In other words, if our condition number is small and our residual is small compared to \(b\), then we know our solution must be accurate. \(\square\)