DISCUSSION 2 SOLUTIONS

MATH 170A

Problem 0.1. Consider a string of 3 carts separated by springs, and attached to fixed walls by springs. The springs between the cars have spring constant of 1 N/m, while those attached to the walls have spring constant of 5 N/m. If the middle cart is pushed to the right with a force of 5 N, how far from their equilibrium positions will each cart move?

Solution. As in class, the general equation is

\[-k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) = -f_i,\]

where \(k_i\) is the spring constant for the spring to the left of the \(i\)-th car, \(x_i\) is the displacement from equilibrium of the \(i\)-th car, and \(f_i\) is the force on the \(i\)-th car, all measured with the right direction being positive. This simplifies to

\[k_i x_{i-1} - (k_i + k_{i+1}) x_i + k_{i+1} x_{i+1} = -f_i.\]

Using the given spring constants, the matrix equation is

\[
\begin{bmatrix}
-6 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
-5 \\
0
\end{bmatrix}.
\]

Solving that gives \((x_1, x_2, x_3) = (0.5, 3, 0.5)\). (This lines up with our intuition, since we would expect the cars to move to the right, and for the middle one that we’re pushing to move the most.)

Problem 0.2. By hand, find the Cholesky decomposition of the following s.p.d. matrix.

\[A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 13 & 27 \\
3 & 27 & 83
\end{bmatrix}\]

Then use that decomposition to solve

\[A \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
8 \\
28 \\
77
\end{bmatrix}.
\]

(Don’t just use normal row reduction! Use the Cholesky decomposition! Hint: The numbers should work out nicely.)
Solution. I will do this by the erasure method, meaning I'll start with $A$, then calculate entries one or more at a time, and erase the former entries.

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 13 & 27 \\
3 & 27 & 83
\end{bmatrix} \Rightarrow
\begin{bmatrix}
1 & 2 & 3 \\
0 & 13 & 27 \\
0 & 0 & 83
\end{bmatrix} \Rightarrow
\begin{bmatrix}
\sqrt{1} & 2 & 3 \\
0 & 13 & 27 \\
0 & 0 & 83
\end{bmatrix} \Rightarrow
\begin{bmatrix}
1 & \frac{2}{\sqrt{1}} & \frac{3}{\sqrt{1}} \\
0 & 13 & 27 \\
0 & 0 & 83
\end{bmatrix} \Rightarrow
\begin{bmatrix}
1 & \sqrt{1} & \sqrt{1} \\
0 & 3 (27 - 2 \cdot 3)/3 \\
0 & 0 & \sqrt{83 - 7^2 - 3^2}
\end{bmatrix} =
\begin{bmatrix}
1 & 2 & 3 \\
0 & 3 & 7 \\
0 & 0 & 5
\end{bmatrix}
\]

That last matrix is my Cholesky factor. (You can check it by multiplying $R^T R$.)

Next, I want to solve
\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 3 & 0 \\
3 & 7 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
8 \\
28 \\
77
\end{bmatrix}.
\]

I start by solving
\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 3 & 0 \\
3 & 7 & 5
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{bmatrix} =
\begin{bmatrix}
8 \\
28 \\
77
\end{bmatrix},
\]
which gives $(\bar{x}, \bar{y}, \bar{z}) = (8, 4, 5)$. Then I solve
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 3 & 7 \\
0 & 0 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
8 \\
4 \\
5
\end{bmatrix},
\]
which has solution $(x, y, z) = (7, -1, 1)$. (We can check this by multiplying the original $A$ by this vector.) □

**Problem 0.3.** If the Cholesky decomposition algorithm works (i.e., you never have to take the square root of a negative number or zero for the diagonal elements), then the original matrix is s.p.d.. Prove that is true. (Hint: Don’t look so much at the algorithm; look at what the algorithm produces.)

**Solution.** If the algorithm works, it splits a matrix $A$ into $R^T R$, where $R$ is upper triangular, with positive numbers on the diagonal. Since the numbers on the diagonal can’t be zero, and a triangular matrix is nonsingular as long as the diagonal is nonzero, then $R$ is a nonsingular matrix as well.

To see that $A$ is symmetric, we calculate
\[A^T = (R^T R)^T = R^T (R^T)^T = R^T R = A,\]
and so $A$ is symmetric.

To see that $A$ is positive definite, consider
\[x^T A x = x^T R^T R x = (Rx)^T R x.\]
Thus $x^T Ax$ can be interpreted as the dot product of $Rx$ with itself. It is known that the inner product of a vector with itself is positive unless the vector is zero. But, since $R$ is nonsingular, $Rx$ is zero if and only if $x$ is zero. Thus $x^T Ax > 0$ unless $x$ is zero, and so $A$ is positive definite. □