Day 17: 8.1 (Midterm Monday! 6.3-7.7.) (Today's not on Midterm, final) The rest of the term we'll be jumping from topic to topic. Some directly to do with calculus, some not. Today and next two lectures are geometry!

Q1: Let \( f(x) \) represent the height of a shape at \( x \), both measure in cm. What are the units of \( \int_0^2 f(x) \, dx? \)

Units of \( f \), units of \( dx \) = cm, cm = cm\(^2\), area!

\[ \int_0^2 f(x) \, dx \] gives area, if \( f(x) \) is height.

Let's say we wanted to find area of this figure. How? One way: slice into rectangles and add up!

<table>
<thead>
<tr>
<th>f(x)</th>
<th>height</th>
<th>Area of rectangle = base \cdot height</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>dx</td>
<td>[ f(x) , dx ]</td>
</tr>
</tbody>
</table>

Add up all the slices with an integral: \( \int_0^2 f(x) \, dx = \text{area!} \)

Key idea (for 8.1, 8.2): Total area = \( \int \text{area of thin slice}. \)

Q2: \( \int_0^2 \)

Set up integral to calculate area.

Way 1: Slice vertically. For the slice at position \( x \), since similar triangles, we have

\[ \frac{x}{h} = \frac{1}{2} \Rightarrow h = 2x \]

so area of slice is \( 2x \cdot dx \) (\( x \) is location, \( dx \) is (thin) width)

so \( \int_0^1 2x \, dx \) (Note: \( \frac{x^2}{2} \) evaluated from 0 to 1)

Don't have to cut vertically! Way 2:

\[ \frac{y}{w} = \frac{2}{1} \Rightarrow w = \frac{y}{2}, \text{ area of slice is } \]

\( \frac{y}{2} \cdot dy \)

Total area = \( \int_0^2 \frac{y}{2} \, dy \) \((\frac{y^2}{4} \text{ evaluated from 0 to 1})\)

Limits match up with direction of the thickness.

Now, really, we already (generally) know how to find areas. If normal shape, could use normal integration or just a formula. Really, this is just to warm up for 3d figures.

For 3d figures, we essentially do the same thing!

Slice the figure into pieces we can find the volume for, then add up with an integral!

Key idea: Total volume = \( \int \text{volume of thin slice}. \)

ex: Box: \( 3 \times 2 \times 1 \)

Slice: Volume of slice = area \cdot thickness

\[ = 2 \cdot 3 \cdot dx \] (not changing so no \( x \) in integral)

add up with integral:

\[ \text{total volume} = \int_0^1 6 \, dx \] adding up the slices

from \( x = 0 \) to \( x = 1 \). 

Volume of slice = \( 1 \cdot 2 \cdot dx \)

Total = \( \int_0^3 2 \, dx \) adding up slices from \( z = 0 \) to \( z = 1 \).
In this case, it didn't really matter which slices we took.
But often some slices are much easier than others.

**Ex:** Cone:

horizontal slices are much nice: circles!

volume of slice = area of circle • thickness.

**Q3:** Find volume of cone with height = 2, with base radius = 1. (Hint: Find radius in terms of distance from tip)

Need area of slice, so need area of circle. For area of circle, need radius.

\[
\frac{h}{r} = \frac{2}{1} \quad \Rightarrow \quad r = \frac{h}{2}
\]

So volume = \( \pi \left( \frac{h}{2} \right)^2 dh = \frac{\pi}{4} h^2 dh \)

\[
total\ \text{volume} = \int_0^2 \frac{\pi}{4} h^2 dh = \frac{\pi}{4} \left[ \frac{1}{3} h^3 \right]_0^2 = \frac{2}{3} \pi
\]

Any shape we can find volume of thin slices for, we can find volume.

**Ex:**

sphere: Slices are circles.

pyramids: Slices are rectangles.

etc.

8.2. It's essentially some less obvious examples. Two days.
Day 18: 8.2
Finding volumes of shapes: (Review bit)

Idea: 1. Slice your shape into thin slices whose volumes are easy to find.
   2. Use an integral to add them up.

ex: 

\[ \text{volume of slice is } \pi r^2 \, dh \]

need \( r \) is terms of \( h \): \( r = \frac{h}{2} \), so

\[ \text{volume of slice} = \pi \left( \frac{h}{2} \right)^2 \, dh \]

Add up with an integral

\[ \text{total volume} = \int_0^2 \pi \left( \frac{h}{2} \right)^2 \, dh = \frac{2}{3} \pi \]

Today, a similar thing: Find volumes of "solids of revolution"

What is a solid of revolution?

If \( f(x) \) rotate \( f(x) \) around \( x \)-axis, get a shape like a table leg.

Let me show you a picture of a more complicated one.

(Open ADF).

Q1: Talk to your neighbor: How would you slice it to find volume? How to find volume of each slice?

Show cans again: We're going to slice into cone shapes again.

Ex: Take region under \( y = x^3 \), rotated round \( x \)-axis, \( b/w \) \( x=0 \) and \( x=1 \)

Looks like a teak.

Slice into circles

value of slice = area of circle \( \times \) \( dx \)

\[ = \pi r^2 \, dx \]

\[ = \pi \left( x^3 \right)^2 \, dx \]

Total volume = \( \int_{x=0}^{x=1} \pi x^6 \, dx = \pi \left[ \frac{x^7}{7} \right]_0^1 = \frac{\pi}{7} \)

Q2: Take the region \( b/w y=x^3 \), the \( y \)-axis and \( y=1 \) and rotate it around the \( y \)-axis. What integral represents that volume? (include picture of region) (so bowl-like region)

Should slice horizontally.

value of slice = area of circle \( \times \) \( \text{d}y \)

\[ = \pi r^2 \, \text{d}y \]

What is \( r \)? \( x \) value as a function of \( y \): \( y = x^3 \), \( x = y^{1/3} \)

\[ = \pi (y^{1/3})^2 \, \text{d}y = \pi y^{2/3} \, \text{d}y \]

Total volume = \( \int_0^1 \pi y^{2/3} \, \text{d}y = \pi \left[ \frac{y^{5/3}}{5/3} \right]_0^1 = \frac{3}{5} \pi \)
of course, sometimes don't have quite circular slices.

\[ \text{Ex: Take region b/w } y=x^2, \ y=1 \text{ and } y-\text{axis and rotate it around } x-\text{axis. What is volume?} \]

like in graph I showed on the computer, slices are not quite circles. They're circles with middle missing. (Annulus)

What is area? \[ \text{area of big circle} - \text{area of small circle}. \]

\[
\text{volume of slice} = \text{area} \cdot dx = (\pi R^2 - \pi r^2) \, dx \\
= (\pi \left[ 1^2 - \pi \left( \frac{3}{2} \right)^2 \right] \, dx = \pi \left( 1 - x^2 \right) \, dx
\]

\[
\text{total volume} = \int_0^1 \pi \left( 1 - x^2 \right) \, dx = \frac{1}{2} \pi
\]

Q3: Take the region bounded by \( y = x^2 \) and \( y = -x^2 + 2 \), then rotate it around the \( x \)-axis.

Which integral represents this area?

\[ \text{where do they intersect? } x^2 = -x^2 + 2 \Rightarrow 2x^2 = 2 \Rightarrow x = \pm 1 \]

\[
\text{volume of slice} = \text{area} \cdot dx = (\pi R^2 - \pi r^2) \, dx \\
= (\pi \left( -x^2 + 2 \right)^2 - \pi (x^2)^2) \, dx
\]

\[
\text{total volume} = \int_{-1}^1 \pi \left( -x^2 + 2 \right)^2 - \pi (x^2)^2 \, dx
\]

\[
= \pi \int_{-1}^1 (4x^2 + 4 - x^2) \, dx = \pi \int_{-1}^1 (4x^2 + 4) \, dx
\]

\[
= \pi \left[ \frac{4}{3} x^3 + 4x \right]_{-1}^1 = \frac{16}{3} \pi
\]

\[
= \text{annulus} \quad \frac{4 - \frac{8}{3}}{\pi} = \frac{16}{3} \pi
\]
Volumes of regions with known cross sections

Idea: base of 3D figure is a region.

But then the figure comes out of the page.

But rather than rotating, each slice is a known shape, maybe a square, with same side as the height of the region. So smaller squares near edges, bigger in middle.

As always, volume of slice = area of slice \cdot dx

For square = \int s^2 dx.

s is side length, h, height of region.

Let me show you on computer.

Q1: Take region b/w \sin x and -\sin x. Consider a figure with that base, and cross-sections at x = \text{const} that are squares. Which of these integrals represents the volume?

\[
\text{volume of slice} = \text{area of slice} \cdot dx
\]

\[
= \int_0^\pi (2\sin x)^2 dx = 2\pi
\]

If cross-section is different shape, just need to change area of slice.

Ex: If cross-section, equilateral triangles \frac{\pi x}{2}, can use pythagorean to get height, area = \frac{1}{2}bh.

\[
\text{area} = \frac{\sqrt{3}}{4} (2\sin x)^2 = \frac{\sqrt{3}}{2} \sin^2 x
\]

So total volume = \int_0^\pi \frac{\sqrt{3}}{2} \sin^2 x dx = \frac{\sqrt{3}}{2} \pi

Arc length: length of a curve.

For straight line, use pythagorean Thm:

\[
\int_a^b \sqrt{1 + (f'(x))^2} dx
\]

Idea for curve: use pythagorean Thm on small pieces, then add up with an integral.

f(x). If I take a small piece, curve is almost straight, so we can almost use pythag. Thm.

\[
\text{rise} \over \text{run} = \text{slope} = \frac{df}{dx}
\]

Pythagorean says hyp = \sqrt{dx^2 + df^2}

Factor out dx^2 = \int_1 + (f')^2 dx

So total arc length = \int_0^b \sqrt{1 + (f')^2} dx
Q2: Find arc length of \( f(x) = 3x^{3/2} \) from \( x = 0 \) to \( x = 1 \).

\[ f'(x) = \frac{3}{2} \cdot 3x^{1/2} = \sqrt{x} \]

Length = \( \int_0^1 \sqrt{1 + \left( \frac{3}{2} \cdot 3x^{1/2} \right)^2} \, dx \)

Let \( u = 1 + x \), then \( du = dx \)

\[ = \int_0^1 \sqrt{u} \, du \]

\[ = \left[ \frac{2}{3} u^{3/2} \right]_0^1 \]

\[ = \frac{2}{3} \cdot 2^{3/2} - \frac{2}{3} \cdot 1^{3/2} = \frac{2}{3} \left( 2^{3/2} - 1 \right) \approx 1.219 \]

Q3: Set up integral to find arc length of \( y = \sqrt{1 - x^2} \) from \( x = -1 \) to \( x = 1 \).

Half-circle, \( f'(x) = \frac{-2x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}} \)

\[ \int_{-1}^1 \sqrt{1 + \left( \frac{-x}{\sqrt{1-x^2}} \right)^2} \, dx = \int_{-1}^1 \sqrt{1 + \frac{x^2}{1-x^2}} \, dx \]

\[ = \int_{-1}^1 \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}} \, dx = \int_{-1}^1 \sqrt{\frac{1}{1-x^2}} \, dx \]

\[ = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \, dx \]

Which we could solve by trig substitution to get \( \pi \).

Which is what it should be.

In general, though, \( \int_a^b \sqrt{1 + (f'(x))^2} \, dx \) is impossible to calculate by hand. Have to approximate it, with Riemann sum.

(Recall, huh. Area under the curve \( \int f(x) \) represents length of a different curve.)
Differential eqns: Any equation with a derivative in it.

We'll spend 5 days. Huge, important area. Oh well.

Today is an intro, so we'll work through a simple example, doing a few different ideas that we'll look at more closely later.

Ex: If you take a mortgage (literally "death pledge") for $420,000 at 4% interest/year, for 30 years. Your monthly payment is $2000.

Let's set up a differential eqn for this.

Call $P(t) =$ principal (in thousands of dollars), after t years.

rate of change of principal = yearly interest - yearly payment

$$\frac{dP}{dt} = .04 \frac{P}{12.2} - 24$$

- 4% of amount owed
- 12.2 thousand $s.

-explain why it makes sense.

Solving numerically. (Euler's method, 11.3)

At beginning, Principal = 420 (thousand), i.e. $P(0) = 420$.

So rate of charge is $\frac{dP}{dt} = .04 \times 420 - 24 = -7.2$ thousand $\$/year (should be neg > one less each year)

so, if you start owing 420K, after 1 year, one is 412.8K

Q2: Instead of approximating for a year, how much do you still owe after one month?

$\frac{1}{12} \times -7.2$ thousand $\$/year $\Rightarrow -0.6$ thousand $\$/month

so $\approx 420 - 0.6 = 419.4$ thousand $\$$. (so $\approx $600 of your $2000 is paying off your loan, $1400 is interest)

-> okay, how much do you pay off 2nd year? $\frac{dP}{dt}$ is no longer -7.2!

Q2: What is $\frac{dP}{dt}$ at 2 years, using this approach?

$\frac{dP}{dt} = .04 \times 412.8 - 24 \approx -7.49$ thousand $\$/year. Pay off more.

so, after 2 years, one is 412.8 - 7.49 = 405.31 thousand $\$

Okay, will do this more on Friday?

Exact soln: A soln to $\frac{dP}{dt} = .04 \frac{P}{12.2} - 24$ is a function $P(t)$ satisfying the eqn.

will say how to find this on Wednesday, but

$$P(t) = C e^{\frac{t}{25}} + 600$$, where C is some constant.

Q3: If $C = -1$, then $P(t) = 600 - e^{\frac{t}{25}}$. Check this is a soln by plugging it in.

$$\frac{dP}{dt} = - e^{\frac{t}{25}} \times \frac{1}{25} = .04 (600 - e^{\frac{t}{25}}) - 24 = \frac{1}{25} (600 - e^{\frac{t}{25}}) - 24 = \frac{1}{25} \times (600 - e^{\frac{t}{25}}) - 24 = - \frac{1}{25} e^{\frac{t}{25}}$$

Sure are, so equal.


\[ \frac{dp}{dt} = .04p - 24 \] has infinitely many solutions (each depending on how much you owe at the start).

But all the solutions look like \( p(t) = Ce^{t/25} + 600 \).

So, we call the general solution all the solutions together sometimes called a family of solutions.

Interest payment, so never pay it off.

How do you know what \( C \) is?

Need more information than just \( \frac{dp}{dt} = .04p - 24 \).

(Usually) use initial condition \( p(0) \).

Q4: \( p(0) = 420 \). What is \( p(t) \)?

\[ 420 = Ce^{0/25} + 600 \]
\[ 420 = C + 600 \]
\[ C = -180 \]

Particular solution is \( p(t) = -180e^{t/25} + 600 \).

First-order DE: only first derivatives: \( p' = .04p - 24 \).

Second-order DE: up to 2nd derivative:

\[ a = \frac{dv}{dt} = \frac{ds}{dt} \]

So gravity is \( \frac{ds}{dt^2} = -9.8 \, \text{m/s}^2 \).

Both come up a lot. Super important. 2nd more so, but other than acceleration, we'll focus on 1st order.
Last time we talked about a differential eqn.

\[ \frac{dP}{dt} = 0.4P - 24. \]  

The general soln was \( P(t) = Ce^{0.4t} + 600. \)

How did we find that soln?

Usually, hard/possible to find an explicit form for a soln to a differential eqn.

But if it is in a particular form, it's easy (ier).

The basic idea we'll use today is kind of the reverse of something we did with substitution.

```
did this: U = x^2
    du = 2x dx

But if we started with \( \frac{dy}{dx} = 2x \), and wanted to find
    \( \int 2x dx \)
    \( \int 2x dx = x^2 + C \)
```

But if \( \frac{dy}{dx} = 2x \), \( y \) could go like this:

```
\( \frac{dy}{dx} = 2x \)  
\( dy = 2x dx \)  
\( \int dy = \int 2x dx \)
\( y = x^2 + C \)  
```

Let's do this idea for a differential eqn.

Ex: Solve \( \frac{dP}{dt} = P \)  
(simpler version of the earlier ex).

\( dP = P \ dt \)  
\( \int \frac{1}{P} \ dP = \int dt \)  
\( \ln|P| = t + C \)  
\( P = e^{t+C} = e^t e^C = B e^t \)

So the general solution to \( \frac{dP}{dt} = P \) is \( B e^t \).

So, the trick is to get all the \( P\)'s (or \( f\)'s or whatever) to one side, and all the variables to the other side. Then you can integrate and solve.

If you can rewrite \( \frac{df}{dx} \) as \( \int \frac{df}{f} \) or \( \int \frac{dx}{x} \), you can do this.

We call this kind of eqn separable.

Q1: Which of the following are separable? (answer: all but 4th, 1st, 2nd, 3rd)  
\( \frac{df}{dx} = x \)  
\( \frac{df}{dx} = x + 3 \)  
\( \frac{df}{dx} = f + 3 \)  
\( \frac{df}{dx} = x + f \)  
\( \frac{df}{dx} = e^x \)  
\( \frac{df}{dx} = e^x \)  
\( \frac{df}{dx} = e^x \)  
\( \frac{df}{dx} = x \)  
\( \frac{df}{dx} = x dx \)  
\( df = x dx \)  
\( df = x dx \)  

Can solve these one other ways, but not separable.
Q2: Find the general solution for \( y' = -\frac{x}{y} \). (\( y' = \frac{dy}{dx} \))

\[
\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y \, dy = -x \, dx
\]

\[
\int y \, dy = -\int x \, dx
\]

\[
\frac{y^2}{2} = -\frac{x^2}{2} + C_{\text{new constant}}
\]

\[
y^2 = -x^2 + 2C \quad \Rightarrow \quad y^2 + x^2 = C
\]

Family of solutions:

- These are the only kind of diff eqns we'll really do this term, explicitly. 11.5/11.6 rs just applications.

Q3: Find the solution of \( \frac{df}{dx} = \frac{f}{x} \) with \( f(2) = 3 \).

\[
\int \frac{df}{f} = \int \frac{dx}{x}
\]

\[
\ln|f| = \ln|x| + C
\]

\[
|f| = e^{\ln|x| + C} = e^{\ln|x|} e^{C_{\text{new constant}}}
\]

\[
|f| = C|x|
\]

\[
f(x) = \pm C x = C x
\quad \rightarrow \text{Family of solutions}
\]

\[
f(2) = 3 \Rightarrow C \cdot 2 = 3 \quad \Rightarrow \quad C = \frac{3}{2}
\]

\[
f(x) = \frac{3}{2} x
\]
Day 22: 11.3

Last time, we learned how to solve some simple differential equations.

But, for most eqns, you can't solve them explicitly.

ex: \( y' = c \cdot xy \) is not separable. No good way to solve it, explicitly.

But, these more complicated eqns are (usually) more realistic, so even more important.

How to we find solns?

By telling a computer to do it, to approximate its sketch.

How does a computer approximate it?

That's what we're talking about today.

So important, so you have some idea what the computer is doing.

The math 170 A-C is the one year version of this lecture.

We already did this:

Ex: Borrow \( 4200 \) \$, at \( 4\% \). Pay \( 2 \)$/month. df/dt eqn for how much you owe is:

\[
\frac{dp}{dt} = \frac{0.04 \cdot p}{1000} - \frac{24}{12} \text{ thousand $/year.}
\]

To start \( P(0) = 4200 \), so \( \frac{dp}{dt}(0) = 0.04 \cdot 4200 - 24 = -7.2 \text{ thousand dollars/year.} \)

1 year later, owe about \( 4200 - 7.2 = 4122.8 \approx P(1) \)

1 year after that, \( \frac{dp}{dt}(1) = 0.04 \cdot 4122.8 - 24 = -7.49 \text{ new rate of change.} \)

So \( P(2) \approx P(1) - 7.49 = 4045.31 \text{ (estimates).} \)

Could keep going on, or: Do 2 years at once:

\( P(t) = 4200 - \frac{7.2 \cdot 24}{12} = 4056.6 \text{ (estimates).} \)

Idea:

new \( P = \text{old } P + \Delta \text{ input } \times \text{ old } \frac{df}{dt} \) + step size.

\( \Delta \text{ input } \approx \text{ Euler's method: You know the old value, you know how fast it's changing.} \)

Q1: Estimate \( f(4) \) if \( f'(x) = -\frac{3}{x} \) and \( f(2) = 4 \), using \( \Delta \text{ input } = 2 \).

slope at \( x = 0 \) is \( \frac{3}{0} = 0 \).

so \( f'(x) \approx 4 \).

slope at \( x = 2 \) is \( \frac{3}{2} \).

so \( f'(x) \approx 4 + \frac{3}{2} \cdot 2 
\]

\( = 4 + 3 = 7 \text{ so \( f(4) \approx 3.}\) 

This was a bad guess. This one we solved last time: \( f = \text{ circle.} \)

Graphically, this let's us see what Euler's method is.

We are estimating the solution with little line segments.

We use the eqn to determine the slope, linear approximation.

If we reduce the step size, we get a better estimate:

ex: step size = 1, \( \frac{df}{dx} = -\frac{3}{x} \) \( f(0) = 4. \)

\( f(1) \approx f(0) + f'(0) \cdot 1 = 4 - 1.5 = 2.5 \)

\( f(2) \approx f(1) + f'(1) \cdot 1 = 4 + 1.5 \cdot 1 = 5.5 \)

\( f(3) \approx f(2) + f'(2) \cdot 1 = 5.5 + 0.75 \cdot 1 = 6.25 \)

\( f(4) \approx f(3) + f'(3) \cdot 1 = 6.25 + 0.25 \cdot 1 = 6.5 \)

| graph on same one as before. |
Reducing step size reduces error.

Now, I understand this is annoying to do by hand... Though they used to have to.

But let's do one more ex.

Q2: If \( y' = y - x \), \( y(0) = 1 \), estimate \( y(3) \) using \( \Delta x = 1 \).

\[
egin{align*}
  y(1) &= y(0) + y'(0) \cdot 1 = 1 + (0-1) \cdot 1 = 2 \\
  y(2) &= y(1) + y'(1) \cdot 1 = 2 + (2-1) \cdot 1 = 5 \\
  y(3) &= y(2) + y'(2) \cdot 1 = 5 + (5-2) \cdot 1 = 12
\end{align*}
\]

Actual soln: \( y(x) = 2e^{-x} - x - 1 \), so \( y(1) = 2e - 2 \approx 3.4 \)

\[
egin{align*}
  y(2) &= 2e^2 - 3 \approx 11.8 \\
  y(3) &= -4 + 2e^3 \approx 36.2
\end{align*}
\]

It really seems it doesn't do a good job of estimating. But that's because we're doing too large of step size or smaller. (Show Mathematica)

Rules: Reducing step size by half reduces error by \( \frac{1}{2} \).

\[
\text{half the step size, half the error.}
\]
Day 23: 11.5

Next two days are a handful of "real-life" examples of simple separable equations.

Today is exponential growth/decay. Not gonna cover everything. You are expected to be comfortable with it all.

One confusing part is the different growth rates.

Q3: If population is growing continuously at 2% of the current population per year, the diff eqn is

$$\frac{dP}{dt} = 0.02 \cdot P_{\text{current}}$$

$$\frac{dP}{dt}$$ is "absolute" growth rate, like 30,000 people/year or similar.

0.02 is the relative growth rate, the "interest" rate, if you will.

Q1: What is general soln?

$$\frac{dP}{dt} = 0.02 \cdot dt$$

$$\ln P = 0.02t + C$$

$$P = e^{0.02t + C} = e^{0.02t} \cdot e^C = Ce^{0.02t}$$

$$P = Ce^{0.02t}$$

Exponential growth

Q2: If you start with 1 million people, what is population 1 year later?

Not right way: growth rate = 0.02, ln (1 mi) = 20K, so after 1 year, 1 mi + 20 k = 1.02 million.

Wrong! That's Euler's method. A good approximation.

Correct: Use Soln: $$P(0) = 1 \text{ mi} = Ce^{0.02 \cdot 0} = C$$, so $$C = 1$$ (mi)

$$P(t) = 1 \cdot e^{0.02 \cdot 1} = 1.0202$$ million people.

So, after one year, actually add a bit more.

Continuous growth rate = 0.02 = 2%

"Annual" growth rate (APR = annual percentage rate) ≈ 0.02 ≈ 2.02%.

Not a big difference, but bigger & bigger percentages.

Money in banks is similar: continuous interest rate. We've done this.

Q3: You owe $10k00 in credit card debt, which changes at 20% continuous interest.

If you pay the minimum payment of $400/month, which eqn represents the rate of change of how much you owe?

$$\frac{dP}{dt} = 0.2P - 12.4$$

How long to pay off?

How much interest do you pay?

Practice: What is soln?

P(t) = 24 - 14 e^{2t} if you want to practice.

Takes ≈ 2.7 years to pay it off. Pay ≈ $3k interest.

(word to wise: think student loans are expensive? you'd be insane to have this much credit card debt.)

One thing to note: if you owed exactly $24,000, your $$\frac{dP}{dt} = 2.24 \cdot 4.8 = 0$$

so your principal (amount owed) is not changing!

"equilibrium soln" is a constant soln. $$P(t) = 24$$, is that soln here.

(more in a minute)
Newton's law of cooling/heating: The temperature of an object changes at a rate proportional to the difference between its temperature and the temperature of its surroundings (ambient temp). 

\[ \frac{dT}{dt} = K \left( T - T_a \right) \]

Where: 
- \( T \) is the temperature of the object. 
- \( T_a \) is the ambient temperature. 
- \( K \) is a constant. 
- \( t \) is time. 

If \( t \) is measured in hours,

\( K \) is the insulation factor, (T obj - T amb) constant. 

\( K = \frac{1}{10} \) if \( K \) is measured in hours. 

**Example:** You collect a sample of fresh lava, currently at 1000°C, and put it in an insulated container with \( K = \frac{1}{10} \). If \( T(t) = e^{t/10} + 20 \) °C and \( T(0) = 1000 \), then:

\[ e^{t/10} = \frac{T(t) - 20}{C} \]

**Question:** How long till temp of boiling water (100°C)?

**Answer:** \( \approx 25 \) hours, good insulation.

**Equilibrium Solns:**

- Starts at hot, will cool down to \( \approx 20°C \). 
- If cold, will warm up to \( \approx 20°C \). 
- But if exactly \( 20°C \), it will just stay. 
- At \( 20°C \), it is in equilibrium with its surroundings.

**Def:** An equilibrium soln is a constant solution (so the graph is a horizontal line).

- An equilibrium soln is **stable**, if nearby solns get closer.
- An equilibrium soln is **unstable**, if nearby solns get farther away.

**How to find?** Constant means \( \frac{dT}{dt} = 0 \) (not changing). So try to find where \( \frac{dT}{dt} = 0 \).

**Ex:**

\[ \frac{dT}{dt} = -\frac{1}{10} (T - 20) = 0 \]

\[ T - 20 = 0 \]

\[ T = 20 \] is a constant (equilibrium) soln.
Day 24a 11.t: More examples:

Math Modeling: Try to take into account all the different factors you can, guess how they affect your quantities. If it's not accurate, reassess pieces or add more.

**Inflow/outflow:**

**Ex:** There is lead in the water of lake Ontario about 10 nanograms/liter = 10 kg/Km^3 (this is why metric rule)

Assuming the incoming water has no lead, find a diff eqn for the concentration of lead over time. The water inflows at 20.9 Km^3/yr, and assume it's evenly mixed. The lake contains 1600 Km^3 of water.

\[
\frac{dc}{dt} = \frac{\text{inflow of lead}}{\text{total volume}} - \frac{\text{outflow of lead}}{\text{total volume of lake}}. \rightarrow \text{note units are} \quad \frac{\text{kg/yr}}{\text{Km}^3} = \frac{(	ext{kg})}{\text{Km}^3}/\text{yr}.
\]

\[
\text{inflow} = 0. \quad \text{outflow} = \text{concentration} \cdot \text{volume leaving} = C \cdot 20.9
\]

so 
\[
\frac{dc}{dt} = -\frac{20.9}{1600} C
\]

easy to solve.

\[
\frac{dc}{C} = -\frac{20.9}{1600} dt \quad \ln|C| = -\frac{20.9}{1600} t + k
\]

\[
C(t) = K e^{-\frac{20.9}{1600} t} \quad C(0) = 10, \text{ so } C(t) = 10 e^{-20.9t/1600}. \text{ (Exponential decay to 0.)}
\]

Hm... but lead in great lakes not going down anywhere near that fast. Must have lead coming in.

**Q1:** Same example. But now consider inflow of the 20.9 Km^3/yr of inflow, 175 Km^3/yr come from lake Erie. The water in lake Erie has a concentration of about 5 ng/l = 5 kg/Km^3. Assume the rest of the water is lead free, and find the diff eqn for the concentration of lead.

\[
\frac{dc}{dt} = \frac{\text{inflow of lead}}{\text{total volume}} - \frac{\text{outflow of lead}}{\text{total volume}}.
\]

\[
\text{outflow same. } C \cdot 20.9. \quad \text{Inflow} = \text{inflow of water} \cdot \text{concentration} = 175 \cdot 5
\]

\[
\frac{dc}{dt} = \frac{175.5}{1600} - \frac{C \cdot 20.9}{1600} = \frac{175}{1600} - \frac{20.9 C}{1600}.
\]

Solving, get:
\[
C(t) = \frac{5}{20.9} \left( 243 e^{-20.9t/1600} + 175 \right) \quad \text{(practise).}
\]

But that still can't be right.

reaches 5 after just 15 years.

Not going down that fast.

What are other sources of lead? - Actually ask them.

- The rest of the inflow = 34 Km^3/yr.
- Atmospheric deposition.
- Maybe dissolving from mineral sources?

\( \triangleright \) beg one.
Q2: For Lake Ontario, the estimated atmospheric input was
\[ \approx 820 \text{ mg/m}^2/\text{year} = 0.82 \text{ kg/Km}^2/\text{year}. \]
If the surface area of Lake Ontario is \( \approx 19000 \text{ Km}^2 \), find a differential equation for the concentration of lead.
\[ \frac{dc}{dt} = \frac{\text{input of lead}}{\text{total volume}} - \frac{\text{output of lead}}{\text{total volume}}. \]
Output rate = 209 \( \text{mg} \)
Input = \( \text{atmospheric input} + \text{water input} \)
\[ = 8.2 \times 19000 = 15580 \text{ kg/year} \]
\[ \text{woah! That's a ton more than the water input!} \]
Total = 16455
\[ \frac{dc}{dt} = \frac{16455}{1600} - \frac{209}{1600} \]
This is an exponential decay to an equilibrium soln.
Q3: What is the equilibrium soln?
\[ \frac{dc}{dt} = 0 = \frac{16455}{1600} - \frac{209}{1600} \]
\[ \frac{209}{1600} \cdot C = \frac{16455}{1600} \]
\[ C = \frac{16455}{209} = 78.7 \text{ Kg/Km}^3. \]
\[ \text{So, the soln is } C(t) = \frac{5}{209} \left( 3.291 - 283.7 e^{\frac{-209t}{1600}} \right) \]

Hmm, but lead levels are not spiking.
So, there must be something else going on!
What else could be going on? (Ask class)
- hydrolyzed by water (turns into particle, falls to floor) (could measure sediment lead levels)
- attach to other particles, settles (again measure)
- absorbed by fish/plants (measure sediment/fish/poop)
Day 25: 9.2

Questions about lecture glass. Remud-HW due Monday. Do cape evaluations. (at 80%)

Def: A sequence is a list of numbers, like 1, 2, 3, 4, ... etc.

Q1: What is the next # in the sequence 1, 2, 3, 4, ... ?
Answer: Not enough info! 1, 2, 3, 4, 17... is just as good as 1, 2, 3, 4, 5,...
But don’t worry, I’ll always mean the obvious sequence.

Def: A series is an added up sequence: \(1 + 2 + 3 + 4 + \ldots\)

Most series/sequences are complicated, but here’s a simple one:

Def: If each term of a series is the previous one multiplied by the same constant, the series is a geometric series.

ex: \(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\) \((\text{multiply by } \frac{1}{2})\)

ex: \(3 - 6 + 12 - 24 + 48 + \ldots\) \((\text{multiply by } -2)\)

ex: \(\pi + \pi^2 + \pi^3 + \pi^4 + \ldots\) \((\text{multiply by } \pi)\)

ex: \(a + a \cdot x + a \cdot x^2 + a \cdot x^3 + \ldots\) \((\text{multiply by } x)\)

Q2: Can be finite length or infinite length.

There’s a standard trick for adding these up.

ex: \(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} = S\) \((\text{for sum})\)

Want to find \(S\). Trick: multiply both sides by the ratio of terms, \((\frac{1}{2}, \text{here})\)

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} = \frac{1}{2} S
\]

most of the terms overlap, subtract em!

\[
S - \frac{1}{2} S = 1 + 0 + 0 + \ldots + 0 + 0 = \frac{1}{1024} = \frac{2047}{2048}
\]

\[
\frac{1}{2} S = \frac{2047}{2048}, \quad S = \frac{2047}{1024}, \quad \frac{1}{2} S = \frac{2047}{2048}, \quad S = \frac{2047}{1024}
\]

More generally:

\[
S = a + a \cdot x + a \cdot x^2 + \ldots + a \cdot x^{n-1}
\]

\[
XS = a \cdot x + a \cdot x^2 + \ldots + a \cdot x^n + a \cdot x^n
\]

\[
(1-x) S = S - XS = a - a \cdot x^n \quad \Rightarrow \quad S = a \frac{1 - x^n}{1 - x}
\]

Q3: What is \(1 + \frac{1}{2} + \frac{1}{4} + \ldots\) ?

- What if infinitely many terms?
- What does it even mean to add up infinitely many things?

Call \(S_n\) the sum of the first \(n\) terms. \((\text{Partial sum})\)

Def: The sum of an infinite series is defined to be the limit of the partial sums as \(n \to \infty\). \(\Rightarrow \quad \lim_{n \to \infty} S_n\)

More colloquially – add up more and more terms and see if it converges.

Q3: What is \(1 + \frac{1}{2} + \frac{1}{4} + \ldots\) ?
From before \( S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \) = \( \frac{1}{1 - \frac{1}{2}} \),

so \( \frac{1}{1 - \frac{1}{2}} = 2 \). \( \checkmark \)

more generally, the sum of an infinite geometric series

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}
\]

goes to 0, as \( n \to \infty \).

Q4: When does \( x^n \to 0 \)? (What condition on \( x \) do I need?)

need \( |x| < 1 \). If \( x = 3 \), \( 3^n \to \infty \). But if \( x = -\frac{1}{2} \), \( (-\frac{1}{2})^n \to 0 \).

Thm: The sum \( a + ax + ax^2 + \cdots = \frac{a}{1-x} \). if \( |x| < 1 \).

Q5: Calculate \( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \) etc.

\[
\frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = \frac{2}{3}.
\]

Ex: \( 0.9 = 1 \).

\[
\frac{0}{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots
\]

\[
= \frac{0.9}{1 - \frac{1}{10}} \quad (a = \frac{9}{10}, \quad x = \frac{1}{10})
\]

\[
= \frac{0.9}{\frac{9}{10}} = 1. \quad \checkmark
\]
Review: linear approximation/tangent line.

For various reasons, it’s useful to be able to approximate functions with simpler ones. One simple way was linear approximation, which we did last term.

How did we find the line? It was the line with the same value at $x=a$ and the same slope at $x=a$, as $f(x)$.

Q1: Find linear approximation of $e^x$ at $x=0$.

$f(x) = e^x$, $f(0) = e^0 = 1$
$f'(x) = e^x$, $f'(0) = 1$

so $y = mx + b$

$\frac{f(x)}{f'(x)}$ linear approx.

You may remember formula: $f(x) \approx f(a) + f'(a)(x-a)$

This is line w/ slope $f'(a)$ going through the point $(x, y) = (a, f(a))$.

Line is good. But still not amazing. Better would be if I approximated with a quadratic. What? How?

Approximate $f(x) = e^x$ near $x=0$ with a quadratic polynomial.

Goal: Want same value, 1st deriv and 2nd deriv at $x=0$.

$f(x) = e^x$
$f'(x) = e^x$ so $f'(0) 	o 1$.
$f''(x) = e^x$

$e^x \approx 1 + x + \frac{1}{2}x^2$

linear approximation

So $e^x = 1 + x + \frac{1}{2}x^2$

Def: The Taylor Polynomial of degree $n$ approximating $f(x)$ at $x=a$ is the polynomial of degree $n$ that has the same value, and the same derivatives up to degree $n$ at $x=a$.

So: $e^x \approx 1 + x \leftarrow$ linear approx $\Rightarrow$ Taylor degree 1 Taylor poly at $x=0$.

$e^x \approx 1 + x + \frac{1}{2}x^2 \leftarrow$ quadratic approx $\Rightarrow$ degree 2 Taylor poly at $x=0$.

Q2: What is degree 3 Taylor poly at $x=0$ approx. $e^x$?

$P(x) = a + bx + cx^2 + dx^3$
$P'(x) = a + 2bx + 3cx^2 + 4dx^3$
$P''(x) = 2b + 6cx + 12dx^2$
$P'''(x) = 6c + 24dx$

$p''(x) = 3 \cdot 2 \cdot 1 \cdot d = 3! \cdot d$

The earlier ones will be the same.

The derivative at $x=0$ just need $P''(0) = f''(0) = 1$.

So $d = \frac{1}{3!} \cdot 1 = \frac{1}{6}$.
So \[ p(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3. \]

What about higher degree? To find the coefficient of \( x^7 \), say, we need to take derivative times.

So \[ g(x) = 7.6.5.4.3.2.1 \cdot 7! \cdot C_7, \]

so \[ C_7 = \frac{g(0)}{7!} \]

Rule: The Taylor poly. of degree \( n \) at \( x = 0 \) is

\[ f(x) \approx p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots + \frac{f^{(n)}(0)}{n!} x^n \]

Centered at \( x = a \) is very similar:

\[ f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!} (x-a)^n. \]

The book has some nice pictures of this.

Why would we care? This is covered somewhat later, but I wanted to mention.

Why cool?

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \]

\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \]

In fact, by plugging in, you can see \( e^{ix} = \cos x + i \sin x \):

\[ e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \ldots \]

So \( e^{i\pi} = \cos \pi + i \sin \pi = -1 \)

So \( e^{i\pi} = -1 \).

Why weird? But super important.

Exponentials and sines are really the same thing!!

Q3: Find Taylor poly of \( \sqrt{x} \) at \( x = 1 \) of degree = 2.

\[ f(x) = \sqrt{x} \]
\[ f'(x) = \frac{1}{2} x^{-1/2} \]
\[ f''(x) = -\frac{1}{4} x^{-3/2} \]

So \[ p_2(x) = 1 + \frac{1}{2} (x-1) - \frac{1}{2} (x-1)^2 \]

\[ = 1 + \frac{1}{2} (x-1) - \frac{1}{8} (x-1)^2 \]
Review

Office hours: MTW, 8-11 a.m.
Final wed at 11:30 - 16 questions, 8 new, 8 old. (seems long, but may be short/straightforward. TAs thought okay)
New seats, 1 note card, bring ID.

Calculations:
1. What is idea of definition of \( \int_a^b f(x) \, dx \)?
   The definition is the area under the curve. (More technically, it is
   \[ \lim_{n \to \infty} \sum_{i=0}^n f(x_i) \Delta x. \]

2. What is the 1st fund. Thm of calculus? If \( F(x) \) is an antiderivative of \( f(x) \),
   \[ \int_a^b f(x) \, dx = F(b) - F(a); \]
   or \( \int_a^b f(x) \, dx = F(b) - F(a) \) i.e., Area under rate of change curve gives total change in original function.

3. Calculate \( \frac{d}{dx} \int_0^x \sin(\pi t^2) \, dt \)
   2nd fund. Thm: \( \int_a^x f(t) \, dt \) is an antiderivative of \( f(x) \).
   In other words \( \frac{d}{dx} \left( \int_a^x f(t) \, dt \right) = f(x) \) (derivative of antiderivative is original)
   So \( \frac{d}{dx} \int_0^x \sin(\pi t^2) \, dt = \sin(x \pi x) \).
   Remember, this is how we know there is an antiderivative.

4. Explain why area under curve gives distance traveled?
   Estimate area with rectangles. The area of each is height \( \times \) width, so
   \[ \approx v(t) \Delta t \]
   = Speed \( \times \) Time = Distance traveled.
   Even units check out: \( v(t) \Delta t \) then \( \frac{x}{3} \cdot 5 = 10 \).

5. Explain why \( \int f(x) \, dx = F(x) + C \) has a + C.
   \( \int f(x) \, dx \) is the antiderivative of \( f(x) \),
   But the antiderivative is not unique!
   \[ \text{ex: } f(x) = x^2, \quad F(x) = \frac{x^3}{3} \quad \text{or} \quad \frac{x^3}{3} + 7 \quad \text{or} \quad \frac{x^2}{3} - \pi. \]
   The + C is there so that we include all the possible antiderivatives.
   Sometimes important (differential eqns), sometimes not.

6. Graph the antideriv of
   \[ -2 \]
   Slope is constant -2, so
\( \int x^3 e^x \, dx \) - IBP \implies multiplied

\( \int \left( \frac{1}{\sqrt{x}} - x^{-\frac{3}{2}} \right) \, dx \) → basic → \( x^{-\frac{1}{2}} \)

\( \int \frac{1}{\sqrt{1-x^2}} \, dx \) → trig substitution

\( \int \frac{1}{1-x^2} \, dx \) → partial fractions

\( \int \frac{1}{\ln x} \, dx \) → substitution \( u = \ln x \), \( du = \frac{1}{x} \, dx \)

\( \int \ln x \, dx \) → IBP: \( u = \ln x \), \( v = x \)

Let \( f(t) \) be the rate oil is after the leak begins. What do \( f(3) \), \( \int_0^3 f(t) \, dt \)

\( f(3) = \) rate oil is leaking
\( \int_0^3 f(t) \, dt = \) total oil leakage of first 3 seconds
\( \frac{1}{3} \int_0^3 f(t) \, dt = \) average rate of oil leakage over first 3 seconds