Math 10B - Discussion 4 Solutions

We want to show that the derivative of \( \int_a^x f(t)\,dt \) is \( f(x) \). We define \( F(x) = \int_a^x f(t)\,dt \). Then, for some small change in \( x \), denoted by \( \Delta x \),

\[
F(x + \Delta x) - F(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt
\]

\[
= \int_x^{x+\Delta x} f(t)\,dt.
\]

For some value \( c \) (which depends on \( \Delta x \)) between \( x \) and \( x + \Delta x \), we can calculate the area of the integral \( \int_x^{x+\Delta x} f(t)\,dt \) as \( f(c) \cdot \Delta x \) (height times width). So,

\[
F(x + \Delta x) - F(x) = f(c) \cdot \Delta x
\]

\[
\frac{F(x + \Delta x) - F(x)}{\Delta x} = f(c)
\]

\[
\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} f(c).
\]

We see that the left hand side is, by definition, \( F'(x) = \frac{d}{dx} \int_a^x f(t)\,dt \) while the right hand side goes to \( f(x) \) (since (1) the interval of \( x \) to \( x + \Delta x \) shrinks to only consisting of the value \( x \) and (2) \( c \) must still be in this interval). Hence, \( \frac{d}{dx} \int_a^x f(t)\,dt = f(x) \) as desired.