BAKING THE OPTIMAL LAYER CAKE

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Problem 1. *Suppose we want to make a layer cake inside a cone of height H and radius R. What is the optimal volume that can be taken up by 3 layers? How much of the cone does it occupy? And what is its generalization to N layers?*

This turns out to be an interesting trip involving a little bit of calculus and a lot of recurrence relations.

1. The Case of One Layer

First, let's look at the case of one layer, $N = 1$. This will be, of course, the base case in our recurrence relations. The solution to this problem is often given as an exercise in one-variable calculus (I've given similar problems to my students). Let's denote the height of the single cake layer (a cylinder) as h and its radius as r. Then we have the familiar formula

$$V = \pi r^2 h \tag{1}$$

Because our cylinder is constrained to be inside a cone, we use similar triangles to find the relationship between the height and radius of the cylinder. The top of the cylinder forms the base of a new, smaller cone (this will be a crucial observation in the rest of this problem!) of radius r, and height $H - h$. Because the smaller cone is the top portion of the larger cone, it is similar (its dimensions in the same proportions as) the large cone:

$$\frac{H-h}{r} = \frac{H}{R}$$

This gives $r = \frac{R}{H}(H-h)$. Therefore, we have the volume of the layer can be viewed as a function of just one variable, h, and

$$V(h) = \pi \frac{R^2}{H^2}(H-h)^2 h.$$

To optimize this, we take its derivative, using the product and chain rules, and set it to 0:

$$V'(h) = \pi \frac{R^2}{H^2}(-2(H-h)h + (H-h)^2) = 0$$

if and only if $-2h + (H-h) = 0$, by canceling one factor of $H - h$ (this of course requires that $H - h \neq 0$, but the only time when $H = h$ is when the radius is 0 and the layer is infinitely skinny, a situation only desired by mathematicians who are on a diet). Rearranging, we have

$$-3h + H = 0$$

or

$$h = \tfrac{1}{3}H.$$

This is an interesting result in itself: the optimal way to fit one cylinder inside a cone is for the cylinder to have exactly $\frac{1}{3}$ the height of the cone (and $\frac{2}{3}$ the radius as we'll derive right now). The radius is then

$$r = \frac{R}{H}(H - h) = \frac{R}{H}(H - \tfrac{1}{3}H) = \tfrac{2}{3}R.$$

Note that the coefficients $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$ which multiply the height and radius of the whole cone are in fact independent of the choice of scale. Regardless of whether we have a very tall skinny cone, or a wide fat cone, the optimal cylinder (layer cake) that fits in the cone is always $\frac{1}{3}$ of the height, and $\frac{2}{3}$ of the radius.

2. A Recursion Formula

Now for the crucial observation that lets us generalize to more layers:

Lemma 1. *If we have an N-layer cake which occupies maximum volume in a cone of radius R and height H, then, if we remove the bottom layer (of height h_1 and radius r_1), then the remaining $N - 1$ layers occupy the maximum volume in the smaller cone that sits above the first layer, which has radius r_1 and height $H - h_1$.*

Proof of Lemma. This is a simple observation: if we fix the height h_1 (don't allow it to change) for the moment, then the cone above the first layer has radius r_1 (the layer stops when it hits the side of the cone, so the top disk of the layer is the bottom disk of the cone above). Then, among all configurations of $N - 1$ layers in the cone above, it must already be the optimal configuration. This is because if it were not optimal and the layers could be adjusted, then *since we are fixing h_1*, it yields a configuration of more volume, and once we add the bottom layer, we then have a configuration of more volume for the whole N layer cake.

Now we are ready to at least state the general problem. The general case is a little bit messy, requiring some quantities with multiple indices, but bear with me for this. Let $\alpha_{i,N}$ be the fraction of H that layer i of the cake (numbered from the bottom) takes up in an N-layer cake, and $\beta_{i,N}$ denote the fraction of the radius R taken up by layer i of the cake. These shall henceforth be known as **layer cake coefficients**. In particular, $\alpha_{1,1}$ is the fraction of the height that a volume-maximizing 1-layer cake will take up; we calculated $\alpha_{1,1} = \frac{1}{3}$ by the case above. Similarly, $\beta_{1,1} = \frac{2}{3}$. These form the base case for our following computations.

Now, if we fix N and have a configuration of cake layers, the actual height (thickness) h_i of each layer is $\alpha_{i,N}H$, the radius r_i is $\beta_{i,N}R$, and the total volume taken up by the layers is

$$V_N = \sum_{i=1}^{N} \pi r_i^2 h_i = \pi R^2 H \sum_{i=1}^{N} (\beta_{i,N})^2 \alpha_{i,N}, \tag{2}$$

after substituting these formulas. Actually, noting the fact that the volume of the whole cone is $\frac{1}{3}\pi R^2 H$, it's convenient to define the *volume ratio*, which is this volume V_N divided by the volume of the cone:

$$v_N = 3 \sum_{i=1} (\beta_{i,N})^2 \alpha_{i,N}.$$

Now, using the lemma above, if we fix h_1, which we'll determine later, then considering the cake layers above the first, they fit into the cone above, and thus, the $N-1$ layers take up maximum volume in this upper cone of radius r_1 and height $H - h_1$. For $i > 1$, h_i is not only the ith layer of the N layer cake in the whole cone, it is *also* the $(i-1)$th layer in the upper, smaller cone. Thus, in terms of our fancy notation and coefficients,

$$h_i = \alpha_{i,N} H = \alpha_{i-1,N-1}(H - h_1)$$

and

$$r_i = \beta_{i,N} R = \beta_{i-1,N-1} r_1.$$

Note that by similar triangles, $r_1 = \frac{R}{H}(H - h_1)$, so that everything is actually expressible in terms of h_1. This allows us to derive a relation on the α's and β's! Namely, for $i > 1$,

$$\alpha_{i,N} = \alpha_{i-1,N-1} \frac{H - h_1}{H} \tag{3}$$

and

$$\beta_{i,N} = \beta_{i-1,N-1} \frac{r_1}{R} = \beta_{i-1,N-1} \frac{H - h_1}{H}. \tag{4}$$

Now, $h_1 = \alpha_{1,N} H$, so $H - h_1 = H - \alpha_{1,N} H = (1 - \alpha_{1,N})H$. With the formula for r_1 above, we find $r_1 = \frac{R}{H}(H - h_1) = R(1 - \alpha_{1,N})$ which gives us

$$\beta_{1,N} = 1 - \alpha_{1,N}.$$

Thus we have the formulas, still for $i > 1$, the recurrences

$$\alpha_{i,N} = \alpha_{i-1,N-1} \beta_{1,N} \tag{5}$$

$$\beta_{i,N} = \beta_{i-1,N-1} \beta_{1,N} \tag{6}$$

$$\beta_{1,N} = 1 - \alpha_{1,N}. \tag{7}$$

Note that all of these formulas do not explicitly involve R or H. It remains to determine $\alpha_{1,N}$. This is where the real work is (ok, well, setting all that notational junk and finding a recursive way of thinking about the problem was definitely tough work, too). More precisely, this is where the calculus will come in. Namely, we should consider everything as a function of h_1: we write, using $h_i = \alpha_{i-1,N-1}(H-h_1)$ instead of $\alpha_{i,N}H$ and $r_i = \beta_{i-1,N-1}r_1$ instead of $\beta_{i,N}R$,

$$\begin{aligned} V(h_1) &= \pi\frac{R^2}{H^2}(H-h_1)^2h_1 + \sum_{i=2}^{N} \pi\beta_{i-1,N-1}^2\frac{R^2}{H^2}(H-h_1)^2\alpha_{i-1,N-1}(H-h_1) \\ &= \pi\frac{R^2}{H^2}\left[(H-h_1)^2h_1 + \sum_{i=2}^{N}\beta_{i-1,N-1}^2\alpha_{i-1,N-1}(H-h_1)^3\right]. \end{aligned}$$

This looks like a mess, but note that, at least with a little bit of index-rewriting, we find the messy sum is equal to $\frac{1}{3}$ of the $(N-1)$th volume ratio:

$$\sum_{i=2}^{N}\beta_{i-1,N-1}^2\alpha_{i-1,N-1} = \sum_{j=1}^{N-1}\beta_{j,N-1}^2\alpha_{j,N-1} = \frac{1}{3}v_{N-1}.$$

In all, we have

$$V(h_1) = \pi\frac{R^2}{H^2}\left[(H-h_1)^2h_1 + \frac{1}{3}v_{N-1}(H-h_1)^3\right]. \tag{8}$$

3. The First Layer: Time for Calculus

What does that formula give us? How do we find the maximum volume? We now maximize V respect to h_1 (because we have already supposed that the rest of the h's were found to be optimal for the upper cone, by induction). This is a calculation we can do just like before:

$$\begin{aligned} V'(h_1) &= \pi\frac{R^2}{H^2}\left[-2(H-h_1)h_1 + (H-h_1)^2 - v_{N-1}(H-h_1)^2\right] \\ &= \pi\frac{R^2}{H^2}(H-h_1)\left[-2h_1 + (1-v_{N-1})(H-h_1)\right]. \end{aligned}$$

This is zero if and only if

$$-2h_1 + (1-v_{N-1})(H-h_1) = 0,$$

(unless $H-h_1$ is 0, but that's not a reasonable solution) which means

$$-2h_1 + H - h_1 - v_{N-1}H + v_{N-1}h_1 = 0$$

or

$$(v_{N-1}-3)h_1 + (1-v_{N-1})H = 0.$$

This finally follows if

$$h_1 = \frac{1-v_{N-1}}{3-v_{N-1}}H.$$

In particular, we have found

$$\alpha_{1,N} = \frac{1-v_{N-1}}{3-v_{N-1}} = \frac{1-3\sum_{i=1}^{N-1}\beta_{i,N-1}^2\alpha_{i,N-1}}{3-3\sum_{i=1}^{N-1}\beta_{i,N-1}^2\alpha_{i,N-1}}. \tag{9}$$

Practically, this means, the work in the recurrence relation is to first determine $\alpha_{1,N}$, which is a fancy combination of all the coefficients in the case for $N-1$ layers. Then $\beta_{1,N}$ is just $1-\alpha_{1,N}$, and $\alpha_{i,N}$ and $\beta_{i,N}$ are all simply the previous coefficients multiplied by $\beta_{1,N}$. □

4. The Case $N = 3$

That's all well and good, but what about some concrete computations? As noted already, $\alpha_{1,1} = \frac{1}{3}$ and $\beta_{1,1} = \frac{2}{3}$. Let's calculate $\alpha_{i,2}$ and $\beta_{i,2}$. We have the volume ratio v_1 is $3\left(\frac{2}{3}\right)^2\frac{1}{3} = \frac{4}{9}$. So the single layer takes up $\frac{4}{9}$ of the height. Then

$$\alpha_{1,2} = \frac{1-\frac{4}{9}}{3-\frac{4}{9}} = \frac{\frac{5}{9}}{\frac{23}{9}} = \frac{5}{23}$$

and so

$$\beta_{1,2} = 1 - \frac{5}{23} = \frac{18}{23}$$

Next, we have

$$\alpha_{2,2} = \alpha_{1,1}\beta_{1,2} = \frac{1}{3}\frac{18}{23} = \frac{6}{23}$$

and

$$\beta_{2,2} = \beta_{1,1}\beta_{1,2} = \frac{2}{3}\frac{18}{23} = \frac{12}{23}.$$

Armed with this, we embark on calculating

$$v_2 = \frac{3\cdot 18^2\cdot 5}{23^3} + \frac{3\cdot 12^2\cdot 6}{23^3} = \frac{324}{529}.$$

As you can imagine, since $\alpha_{1,3}$ involves fractions with v_2, we're going to get some big numbers, very fast:

$$\alpha_{1,3} = \frac{1-v_2}{3-v_2} = \frac{205}{1263} \approx 0.1623$$

and

$$\beta_{1,3} = 1-\alpha_{1,3} = \frac{1058}{1263}.$$

Then

$$\alpha_{2,3} = \alpha_{1,2}\beta_{1,3} = \frac{230}{1263} \approx 0.1821 \tag{10}$$

$$\beta_{2,3} = \beta_{1,2}\beta_{1,3} = \frac{276}{421} \approx 0.6556 \tag{11}$$

$$\alpha_{3,3} = \alpha_{2,2}\beta_{1,3} = \frac{92}{421} \approx 0.2185 \tag{12}$$

$$\beta_{3,3} = \beta_{2,2}\beta_{1,3} = \frac{184}{421} \approx 0.4371. \tag{13}$$

The solution to the problem as stated is therefore that the first layer has thickness $\alpha_{1,3}H$, the second, $\alpha_{2,3}H$, and the third is $\alpha_{3,3}H$. The amount of the cone these layers take up, by volume, is v_3, or about 70.172%.

5. Conclusion

Whew! That was a lot of work! I didn't think it'd be this hard! Not intuitive at all! Perhaps we worked way too hard to get our result. One suggestion that seemed to occur to me right away, and I suspect lots of people will submit this solution without giving it another thought, is the following very elegant-seeming method: Find the maximum volume of one layer, and then find the maximum volume of another single layer in the cone above the first layer, and then find the layer of maximum volume in the cone above the second layer, etc., until you get as many layers as you want, N. It's a recursive solution, but much simpler. It has a very elegant solution, as it is simply a geometric series: the second layer is maximal cylinder in a cone of exactly $\frac{2}{3}$ the height, the third layer is in a cone $\frac{4}{9}$ the size, etc. So, since volume scales as the cube of the side lengths, the volumes of each additional layer decrease by a factor of $\frac{8}{27}$.

Since the initial volume ratio of such a thing is $\frac{4}{9}$, a 3-layer cake would occupy a total of

$$\frac{4}{9}\left(1+\frac{8}{27}+\frac{64}{729}\right)=\frac{4036}{6561}\approx 61.515\%,$$

which is clearly less than the 70.172% that our optimal solution occupies.

Anyway, it is illustrative of calculus concepts to see what really goes wrong. Say that you wedge in 2 cylinders using this method, into a cone. Let us ask ourselves what happens when we change the height h_1 of the first cylinder without changing where the second cylinder hits. What this amounts to doing is making the bottom layer change both its height and radius, because the first layer's top circle always has to stay stuck to the cone. *However*, the second cylinder does *not* change in radius. Its height simply gets longer as h_1 decreases, and by a constant rate. So if you imagine changing h_1 by an (negative) infinitesimal amount dh (I might have just made Weierstrass turn in his grave), we have the volume of the second cylinder changes by

$$dV_2 = V(r_2, h_2 - dh) - V(r_2, h_2) = \pi r_2^2(h_2 - dh) - \pi r_2^2 h_2 = -\pi r_2^2 dh$$

which is a positive quantity (since we're taking dh negative). On the other hand, the infinitesimal change in V_1 is 0, because r_1 depends on h_1, and $dV_1 = V'(h_1)dh = 0$ since h_1 is known to be the optimal height for the first layer. The upshot is, decreasing h_1 will *increase* the total volume of the whole assembly, and thus, an assembly of two layers optimal for their particular cones is not optimal. In our solution, *none* of the layers, except the top layer, is optimal for the cone with a base that is the top of the previous layer.

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