

Supplementary Numerical Results for Projected-Search Methods for Bound-Constrained Optimization

Michael W. Ferry* Philip E. Gill† Elizabeth Wong† Minxin Zhang†

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Abstract

This document provides detailed information of the numerical results used to compile the performance profiles in the article “A Class of Projected-Search Methods for Bound-Constrained Optimization” [3].

Key words. Bound-constrained optimization, projected-search methods, line search methods, projected gradient methods, quasi-Newton methods.

1. Introduction

All testing was done on problems taken from the CUTEst test collection (see Bongartz et al. [1] and Gould, Orban and Toint [4]). As of July 1, 2020, the CUTEst test set contains 154 bound-constrained problems of the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad x \in \Omega, \quad (\text{BC})$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a twice-continuously differentiable function and $\Omega = \{x \in \mathbb{R}^n : \ell \leq x \leq u\}$ for vectors of lower and upper bounds such that $\ell \leq u$. Although many problems allow for the number of variables and constraints to be adjusted in the SIF data file, our tests used the default dimensions set in the CUTEst distribution. This gave problems ranging in size from BQ1VAR (one variable) to WALL100 (149624 variables).

The methods tested were the limited-memory reduced Hessian methods LRHB-qWolfe and LRHB-qArmi jo, and the limited-memory method LBFGS-B (Byrd et al. [2], Zhu et al. [6], and Morales and Nocedal [5]). All three solvers were terminated at the first point x_T such that

- (a) $\|P_{x_T}(-\nabla f(x_T))\|_\infty \leq 10^{-5}(1 + |f(x_T)|)$ and
- (b) $|f(x_T) - f(x_{T-1})| \leq 10^7 \epsilon_M \times \max\{|f(x_T)|, |f(x_{T-1})|, 1\}$; or
- (c) $\|P_{x_T}(-\nabla f(x_T))\|_\infty < \sqrt{\epsilon_M}$,

where ϵ_M is the machine precision. In the first iteration of the algorithms, condition (b) is ignored. A time limit of 3600 seconds and an iteration limit of 1000000 was imposed in each case.

*NVIDIA Corporation, Hillsboro, Oregon (michael@mwferry.com).

†Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112 (pgill@ucsd.edu, elwong@ucsd.edu, miz151@ucsd.edu) Research supported in part by National Science Foundation grants DMS-0915220 and DMS-1318480. The content is solely the responsibility of the authors and does not necessarily represent the official views of the funding agencies.

2. The Results

Table 2 gives details of the runs for LRHB-qWolfe, LRHB-qArmijo, and LBFGS-B. The top row describes the results presented in each column. The name and dimension of the CUTEst problem appear in the first column. The second column lists the solver associated with the statistics of the row. Subsequent columns list the various statistics from the run. See Table 1 for a brief description of the entries for each column.

n	Number of variables.
x_T	First point satisfying (a)–(c).
nItn	Number of iterations.
nF	Function evaluations.
$\mathcal{A}(x_T)$	Number of indices in the active set at x_T .
Obj	The value of $f(x_T)$.
Grad	The value of $\ P_{x_T}(-\nabla f(x_T))\ _\infty$.
a	Abnormal exit because of numerical difficulties.
i	Iteration limit exceeded.
t	Time limit exceeded.

Table 1: Notation in tables of results.

Table 2: Final results and statistics from three solvers on 154 problems.

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
3PK $n = 30$	LRHB-qWolfe	60080	62738	0	1.720122914E+00	2.61E-05
	LRHB-qArmijo	78034	89279	0	1.720119390E+00	2.14E-05
	LBFGSB	61426	66529	0	1.720124356E+00	2.61E-05
AIRCRFTB $n = 8$	LRHB-qWolfe	52	61	3	2.696768571E-15	3.69E-06
	LRHB-qArmijo	53	69	3	8.393302208E-16	1.35E-06
	LBFGSB	121	132	3	1.716989867E-12	9.14E-06
ALLINIT $n = 4$	LRHB-qWolfe	11	20	1	1.670596843E+01	1.39E-06
	LRHB-qArmijo	17	24	1	1.670596843E+01	5.48E-06
	LBFGSB	13	16	1	1.670596843E+01	4.19E-05
BDEXP $n = 5000$	LRHB-qWolfe	24	25	0	1.406376688E-06	8.13E-09
	LRHB-qArmijo	24	25	0	1.406376688E-06	8.13E-09
	LBFGSB	25	27	0	2.918591567E-06	9.72E-09
BIGGS3 $n = 6$	LRHB-qWolfe	18	21	3	1.597504093E-12	2.34E-06
	LRHB-qArmijo	19	23	3	1.361473345E-12	2.32E-06
	LBFGSB	26	30	3	8.573885436E-14	3.27E-07
BIGGS5 $n = 6$	LRHB-qWolfe	57	68	1	5.655649810E-03	7.31E-08
	LRHB-qArmijo	60	75	1	5.655649811E-03	7.90E-07
	LBFGSB	112	148	1	5.655649813E-03	3.99E-06
BIGGSB1 $n = 5000$	LRHB-qWolfe	7561	7576	2	1.502691005E-02	5.26E-06
	LRHB-qArmijo	7300	7481	2	1.505144587E-02	8.82E-06
	LBFGSB	8622	8898	2	1.502314065E-02	5.00E-06
BLEACHNG $n = 17$	LRHB-qWolfe	5	6	13	9.176757802E+03	1.00E-06
	LRHB-qArmijo	-- ^a	-- ^a	--	--	--
	LBFGSB	8	30	13	9.176757802E+03	4.13E-05

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
BOX2 $n = 3$	LRHB-qWolfe	15	18	1	1.312926246E-13	5.16E-07
	LRHB-qArmijo	15	17	1	2.201281424E-11	1.26E-06
	LBFGBS	10	13	1	8.452805461E-12	1.84E-06
BQP1VAR $n = 1$	LRHB-qWolfe	1	2	1	0.000000000E+00	0.00E+00
	LRHB-qArmijo	3	7	1	0.000000000E+00	0.00E+00
	LBFGBS	1	2	1	0.000000000E+00	0.00E+00
BQPGABIM $n = 50$	LRHB-qWolfe	23	28	14	-3.790343185E-05	5.59E-06
	LRHB-qArmijo	23	31	14	-3.790343208E-05	5.10E-06
	LBFGBS	20	23	14	-3.790342974E-05	9.54E-06
BQPGASIM $n = 50$	LRHB-qWolfe	21	26	10	-5.519813872E-05	9.40E-06
	LRHB-qArmijo	22	28	10	-5.519813949E-05	7.53E-06
	LBFGBS	23	26	10	-5.519813870E-05	5.87E-06
BQPGAUSS $n = 2003$	LRHB-qWolfe	13735	14037	94	-3.625778071E-01	1.23E-05
	LRHB-qArmijo	-- ^a	-- ^a	--	--	--
	LBFGBS	-- ^a	-- ^a	--	--	--
BRATU1D $n = 5003$	LRHB-qWolfe	39897 ^u	40262 ^u	2	-3.030264570E+22	2.99E+22
	LRHB-qArmijo	-- ^a	-- ^a	--	--	--
	LBFGBS	-- ^a	-- ^a	--	--	--
CAMEL6 $n = 2$	LRHB-qWolfe	9	15	0	-1.031628453E+00	2.66E-08
	LRHB-qArmijo	10	13	0	-1.031628453E+00	3.10E-07
	LBFGBS	11	15	0	-1.031628453E+00	1.62E-08
CHARDISO $n = 2000$	LRHB-qWolfe	2	4	0	5.940231595E-22	1.30E-09
	LRHB-qArmijo	2	4	0	7.868961276E-20	1.07E-08
	LBFGBS	2	4	0	1.300616970E-22	8.43E-10
CHEBYQAD $n = 100$	LRHB-qWolfe	900	911	0	8.716781357E-03	8.00E-06
	LRHB-qArmijo	867	911	0	8.716781709E-03	9.67E-06
	LBFGBS	-- ⁱ	-- ⁱ	--	--	--
CHENHARK $n = 5000$	LRHB-qWolfe	2288	2291	2000	-1.999987169E+00	2.91E-05
	LRHB-qArmijo	1393	1432	2000	-1.999968187E+00	2.59E-05
	LBFGBS	1996	2064	2000	-1.999983585E+00	1.02E-05
CLPLATEA $n = 5041$	LRHB-qWolfe	708	714	70	-1.259207407E-02	8.46E-06
	LRHB-qArmijo	787	819	70	-1.259204541E-02	8.98E-06
	LBFGBS	663	688	71	-1.259202545E-02	8.54E-06
CLPLATEB $n = 5041$	LRHB-qWolfe	291	292	70	-5.094738673E-03	8.80E-06
	LRHB-qArmijo	278	287	70	-5.094740655E-03	5.28E-06
	LBFGBS	280	289	71	-5.094749642E-03	4.37E-06
CLPLATEC $n = 5041$	LRHB-qWolfe	39210	39327	70	-5.020672850E-03	7.88E-06
	LRHB-qArmijo	53531	55312	70	-5.020467259E-03	9.75E-06
	LBFGBS	31164	32255	71	-5.020293290E-03	9.42E-06
CVXBQP1 $n = 10000$	LRHB-qWolfe	9	10	9999	2.250225047E+06	1.35E+00
	LRHB-qArmijo	29	48	9999	2.250225399E+06	2.15E+00
	LBFGBS	1	2	10000	2.250225000E+06	0.00E+00
CYCLOOCTLS $n = 30000$	LRHB-qWolfe	-- ^t	-- ^t	--	--	--
	LRHB-qArmijo	-- ^t	-- ^t	--	--	--
	LBFGBS	-- ^t	-- ^t	--	--	--
DECONVB	LRHB-qWolfe	125	138	3	1.374843746E-08	5.82E-06

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
$n = 63$	LRHB-qArmijo	95	116	1	1.149941322E-08	9.80E-06
	LFGS	105	111	14	1.136515875E-08	9.04E-06
DECONVU $n = 63$	LRHB-qWolfe	123	125	0	2.279536376E-07	5.33E-06
	LRHB-qArmijo	124	141	0	4.638200024E-08	5.16E-06
	LFGS	224	239	12	5.850347886E-08	8.77E-06
DEGDIAG $n = 100001$	LRHB-qWolfe	1	2	100000	1.666658333E+04	0.00E+00
	LRHB-qArmijo	1	2	100000	1.666658333E+04	0.00E+00
	LFGS	2	6	100001	1.666658333E+04	0.00E+00
DEGRID $n = 100001$	LRHB-qWolfe	9	11	0	-9.999949979E+04	4.71E-03
	LRHB-qArmijo	9	11	0	-9.999949979E+04	4.71E-03
	LFGS	8	12	0	-9.999949975E+04	5.24E-03
DEGRID2 $n = 100001$	LRHB-qWolfe	1	2	0	-9.999950000E+04	0.00E+00
	LRHB-qArmijo	4	6	0	-9.999950000E+04	0.00E+00
	LFGS	2	6	100000	-9.999950000E+04	0.00E+00
DIAGIQB $n = 1000$	LRHB-qWolfe	8	15	707	-1.177257582E+15	3.18E+03
	LRHB-qArmijo	35	73	607	-1.211211554E+16	2.20E+08
	LFGS	96	102	706	-1.177256833E+15	3.54E+03
DIAGIQE $n = 1000$	LRHB-qWolfe	154	216	499	-4.546975881E+16	4.62E+04
	LRHB-qArmijo	33	67	3	-7.470000300E+12	6.67E+00
	LFGS	65	75	499	-6.237500406E+14	4.43E+03
DIAGIQT $n = 1000$	LRHB-qWolfe	146	193	293	-3.464270413E+14	4.58E+03
	LRHB-qArmijo	194	219	293	-3.464271079E+14	4.86E+03
	LFGS	93	107	293	-3.464271102E+14	5.68E+03
DIAGNQB $n = 1000$	LRHB-qWolfe	1	6	1000	-3.330837600E+15	0.00E+00
	LRHB-qArmijo	100	111	1000	-3.208478697E+16	0.00E+00
	LFGS	8	23	999	-3.330837600E+15	0.00E+00
DIAGNQE $n = 1000$	LRHB-qWolfe	1	6	1000	-2.502500100E+15	0.00E+00
	LRHB-qArmijo	68	77	1000	-1.758267488E+16	0.00E+00
	LFGS	8	16	999	-2.502500100E+15	0.00E+00
DIAGNQT $n = 1000$	LRHB-qWolfe	1	6	1000	-1.669167600E+15	0.00E+00
	LRHB-qArmijo	86	89	988	-3.944739901E+16	4.63E+07
	LFGS	40	99	999	-1.669167600E+15	0.00E+00
DIAGPQB $n = 1000$	LRHB-qWolfe	5109	5121	0	-8.219669436E+02	7.69E-03
	LRHB-qArmijo	4806	4906	0	-8.219657482E+02	5.59E-03
	LFGS	5338	5513	0	-8.219655346E+02	6.20E-03
DIAGPQE $n = 1000$	LRHB-qWolfe	200	202	0	-3.742735429E+00	3.37E-05
	LRHB-qArmijo	167	176	0	-3.742735429E+00	4.65E-05
	LFGS	185	195	0	-3.742735430E+00	3.66E-05
DIAGPQT $n = 1000$	LRHB-qWolfe	317	319	0	-5.020438948E+02	4.77E-03
	LRHB-qArmijo	257	268	0	-5.020431973E+02	4.76E-03
	LFGS	288	295	0	-5.020439951E+02	3.95E-03
EG1 $n = 3$	LRHB-qWolfe	9	12	1	-1.132800783E+00	4.78E-09
	LRHB-qArmijo	9	14	1	-1.132800783E+00	1.41E-06
	LFGS	9	10	1	-1.132800783E+00	1.94E-06
EXPLIN $n = 1200$	LRHB-qWolfe	234	306	1149	-7.192379779E+07	3.49E+00
	LRHB-qArmijo	93	138	1110	-7.191499563E+07	3.08E+02

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
	LFGS	74	90	1135	-7.192337500E+07	7.05E+00
EXPLIN2 $n = 1200$	LRHB-qWolfe	67	93	1181	-7.199883368E+07	8.60E-01
	LRHB-qArmijo	39	85	1159	-7.199612310E+07	2.25E+02
	LFGS	19	24	1164	-7.199668119E+07	8.94E+00
EXPQUAD $n = 1200$	LRHB-qWolfe	66	82	69	-3.684939176E+09	7.93E+02
	LRHB-qArmijo	20	49	54	-3.675187951E+09	2.31E+04
	LFGS	70	92	75	-3.684939744E+09	3.19E+01
FBRAIN2LS $n = 4$	LRHB-qWolfe	23	29	1	3.683881769E-01	4.49E-07
	LRHB-qArmijo	43	66	1	3.683881769E-01	2.83E-06
	LFGS	43	50	1	3.683881769E-01	1.05E-06
FBRAINLS $n = 2$	LRHB-qWolfe	11	14	0	4.166029455E-01	4.24E-06
	LRHB-qArmijo	11	15	0	4.166029455E-01	6.39E-07
	LFGS	14	20	0	4.166029455E-01	3.70E-06
GENROSEB $n = 500$	LRHB-qWolfe	240	274	499	1.593944932E+03	5.34E-03
	LRHB-qArmijo	379	481	499	1.593944932E+03	4.46E-06
	LFGS	168	175	499	1.593944932E+03	1.45E-07
GRIDGENA $n = 6218$	LRHB-qWolfe	-- ^a	-- ^a	--	--	--
	LRHB-qArmijo	-- ^a	-- ^a	--	--	--
	LFGS	-- ^a	-- ^a	--	--	--
HADAMALS $n = 400$	LRHB-qWolfe	12	31	39	7.311843150E+03	5.33E-06
	LRHB-qArmijo	20	23	39	7.311843150E+03	3.61E-07
	LFGS	10	20	39	7.311843150E+03	4.80E-09
HARKERP2 $n = 1000$	LRHB-qWolfe	40	48	999	-5.000000000E-01	3.60E-12
	LRHB-qArmijo	82	126	999	-5.000000000E-01	1.99E-06
	LFGS	22	31	999	-5.000000000E-01	9.24E-14
HART6 $n = 6$	LRHB-qWolfe	12	26	0	-3.322886892E+00	5.08E-07
	LRHB-qArmijo	15	29	0	-3.322886892E+00	1.22E-05
	LFGS	13	18	0	-3.322886892E+00	2.88E-05
HATFLDA $n = 4$	LRHB-qWolfe	54	72	0	7.676674501E-12	3.30E-06
	LRHB-qArmijo	56	71	0	4.222853796E-11	9.85E-06
	LFGS	29	41	0	2.346412835E-13	5.91E-07
HATFLDB $n = 4$	LRHB-qWolfe	48	66	1	5.572809000E-03	2.59E-07
	LRHB-qArmijo	51	67	1	5.572809001E-03	1.27E-06
	LFGS	21	31	1	5.572809004E-03	4.66E-06
HATFLDC $n = 25$	LRHB-qWolfe	20	23	0	2.499348847E-11	7.16E-06
	LRHB-qArmijo	31	51	0	2.303523969E-11	5.82E-06
	LFGS	19	23	0	8.713517209E-11	7.35E-06
HIMMELP1 $n = 2$	LRHB-qWolfe	11	26	0	-6.205393553E+01	4.66E-08
	LRHB-qArmijo	27	30	0	-6.205393553E+01	1.02E-06
	LFGS	8	13	1	-2.389741895E+01	5.23E-11
HOLMES $n = 180$	LRHB-qWolfe	48	66	176	1.248150348E+03	3.32E-05
	LRHB-qArmijo	92	132	176	1.248150359E+03	1.14E-02
	LFGS	66	72	176	1.248150348E+03	3.15E-03
HS1 $n = 2$	LRHB-qWolfe	19	23	0	4.306836211E-18	7.66E-08
	LRHB-qArmijo	69	113	0	3.707138722E-18	1.59E-08
	LFGS	36	51	0	4.540681526E-14	3.54E-07

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
HS110 $n = 10$	LRHB-qWolfe	6	11	0	-4.577847553E+01	4.99E-10
	LRHB-qArmijo	9	14	0	-4.577847553E+01	8.56E-07
	LBFGB	5	7	0	-4.577847553E+01	2.02E-08
HS2 $n = 2$	LRHB-qWolfe	11	21	1	5.042618789E-02	6.57E-11
	LRHB-qArmijo	10	17	1	4.941229318E+00	2.28E-10
	LBFGB	14	16	1	4.941229318E+00	1.41E-07
HS25 $n = 3$	LRHB-qWolfe	0	1	1	3.283500000E+01	1.99E-08
	LRHB-qArmijo	0	1	1	3.283500000E+01	1.99E-08
	LBFGB	1	2	1	3.283500000E+01	1.99E-08
HS3 $n = 2$	LRHB-qWolfe	3	9	1	1.774937037E-31	2.66E-18
	LRHB-qArmijo	4	10	1	2.835809804E-29	3.37E-17
	LBFGB	3	4	1	1.678585518E-21	2.59E-13
HS38 $n = 4$	LRHB-qWolfe	38	47	0	1.256679185E-15	9.24E-07
	LRHB-qArmijo	129	227	0	2.402442755E-15	1.13E-06
	LBFGB	22	26	0	1.982079454E-14	4.46E-06
HS3MOD $n = 2$	LRHB-qWolfe	4	10	1	2.869859255E-42	3.39E-21
	LRHB-qArmijo	14	26	1	1.203706215E-35	6.94E-18
	LBFGB	5	9	1	1.203706215E-33	6.94E-17
HS4 $n = 2$	LRHB-qWolfe	1	2	2	2.666666664E+00	0.00E+00
	LRHB-qArmijo	2	5	2	2.666666664E+00	0.00E+00
	LBFGB	1	2	2	2.666666664E+00	0.00E+00
HS45 $n = 5$	LRHB-qWolfe	1	4	5	1.000000000E+00	0.00E+00
	LRHB-qArmijo	22	23	5	1.000000000E+00	0.00E+00
	LBFGB	4	11	4	1.000000000E+00	0.00E+00
HS5 $n = 2$	LRHB-qWolfe	6	8	0	-1.913222955E+00	3.54E-06
	LRHB-qArmijo	9	13	0	-1.913222955E+00	6.84E-08
	LBFGB	6	8	0	-1.913222955E+00	2.46E-07
JNLBRNG1 $n = 10000$	LRHB-qWolfe	318	319	3324	-1.805732453E-01	7.64E-06
	LRHB-qArmijo	310	345	3324	-1.805732378E-01	8.22E-06
	LBFGB	286	296	3504	-1.805732277E-01	6.85E-06
JNLBRNG2 $n = 10000$	LRHB-qWolfe	372	374	4058	-4.148652398E+00	2.83E-05
	LRHB-qArmijo	392	403	4058	-4.148652636E+00	4.02E-05
	LBFGB	381	392	4246	-4.148652557E+00	2.71E-05
JNLBRNGA $n = 10000$	LRHB-qWolfe	237	242	3462	-2.711017290E-01	6.47E-06
	LRHB-qArmijo	243	255	3462	-2.711017499E-01	7.84E-06
	LBFGB	238	251	3641	-2.711017099E-01	8.12E-06
JNLBRNGB $n = 10000$	LRHB-qWolfe	1082	1089	4426	-6.300683842E+00	5.96E-05
	LRHB-qArmijo	1237	1275	4426	-6.300685034E+00	7.08E-05
	LBFGB	1263	1305	4617	-6.300685336E+00	3.66E-05
KOEHELB $n = 3$	LRHB-qWolfe	140	209	0	7.751634729E+01	1.83E-06
	LRHB-qArmijo	279	508	0	7.751634729E+01	5.49E-07
	LBFGB	28	36	0	1.122202947E+02	7.17E-05
LINVERSE $n = 1999$	LRHB-qWolfe	187	245	389	6.810001078E+02	5.49E-03
	LRHB-qArmijo	146	195	168	6.810000469E+02	2.52E-03
	LBFGB	103	118	236	6.810000359E+02	1.58E-03
LMINSURF	LRHB-qWolfe	436	438	296	9.000000421E+00	1.35E-05

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
$n = 5625$	LRHB-qArmijo	439	452	296	9.000000246E+00	4.24E-06
	LFBGSE	484	486	296	9.000001529E+00	1.44E-05
LOGROS	LRHB-qWolfe	72	100	0	0.000000000E+00	1.92E-07
$n = 2$	LRHB-qArmijo	184	351	0	0.000000000E+00	8.84E-09
	LFBGSE	80	112	0	1.065814104E-14	1.08E-06
MAXLIKA	LRHB-qWolfe	57	71	4	1.149351476E+03	1.01E-02
$n = 8$	LRHB-qArmijo	206	271	1	1.136307297E+03	1.21E-04
	LFBGSE	192	197	3	1.149350227E+03	5.02E-03
MCCORMCK	LRHB-qWolfe	9	13	1	-4.566580553E+03	2.30E-04
$n = 5000$	LRHB-qArmijo	20	28	1	-4.566580552E+03	1.55E-03
	LFBGSE	10	11	1	-4.566580553E+03	6.71E-04
MDHOLE	LRHB-qWolfe	52	77	1	0.000000000E+00	0.00E+00
$n = 2$	LRHB-qArmijo	120	234	1	6.770847461E-34	5.20E-16
	LFBGSE	61	89	1	3.943461932E-33	1.26E-15
MINSURF	LRHB-qWolfe	17	19	28	1.000000002E+00	1.30E-07
$n = 64$	LRHB-qArmijo	22	33	28	1.000000002E+00	2.88E-07
	LFBGSE	14	17	28	1.000000002E+00	7.38E-07
MINSURFO	LRHB-qWolfe	358	359	460	2.506949517E+00	2.62E-05
$n = 5306$	LRHB-qArmijo	287	294	460	2.506949520E+00	2.88E-05
	LFBGSE	270	271	462	2.506949493E+00	1.17E-05
NCVXBQP1	LRHB-qWolfe	2	6	10000	-1.985543846E+10	0.00E+00
$n = 10000$	LRHB-qArmijo	19	53	9993	-1.985543800E+10	6.66E+02
	LFBGSE	1	2	10000	-1.985543846E+10	0.00E+00
NCVXBQP2	LRHB-qWolfe	26	34	9935	-1.334022594E+10	4.08E+01
$n = 10000$	LRHB-qArmijo	31	59	9934	-1.334022452E+10	2.08E+02
	LFBGSE	74	78	9934	-1.334020435E+10	9.44E+00
NCVXBQP3	LRHB-qWolfe	37	47	9825	-6.558549631E+09	1.67E+02
$n = 10000$	LRHB-qArmijo	938	957	9858	-6.513498290E+09	2.01E+02
	LFBGSE	83	88	9826	-6.557957692E+09	9.50E+00
NLMSURF	LRHB-qWolfe	2708	2718	296	3.894899637E+01	1.18E-04
$n = 5625$	LRHB-qArmijo	2673	2744	296	3.894902982E+01	1.23E-04
	LFBGSE	2638	2704	296	3.894900966E+01	3.66E-04
NOBNDTOR	LRHB-qWolfe	120	122	1154	-4.499331918E-01	6.27E-06
$n = 5476$	LRHB-qArmijo	122	126	1154	-4.499331905E-01	5.18E-06
	LFBGSE	136	142	1158	-4.499332171E-01	4.13E-06
NONSCOMP	LRHB-qWolfe	35	41	0	1.168014480E-11	7.43E-06
$n = 5000$	LRHB-qArmijo	31	35	4	9.820731714E-12	5.71E-06
	LFBGSE	31	37	2	6.443452994E-12	4.63E-06
OBSTCLAE	LRHB-qWolfe	158	159	4865	1.886461223E+00	7.10E-06
$n = 10000$	LRHB-qArmijo	154	157	4865	1.886461228E+00	5.87E-06
	LFBGSE	147	150	4869	1.886461234E+00	9.27E-06
OBSTCLAL	LRHB-qWolfe	91	92	4865	1.886461230E+00	2.44E-05
$n = 10000$	LRHB-qArmijo	95	98	4865	1.886461232E+00	7.22E-06
	LFBGSE	110	111	4868	1.886461243E+00	2.61E-05
OBSTCLBL	LRHB-qWolfe	83	91	2939	7.272155993E+00	8.74E-06
$n = 10000$	LRHB-qArmijo	82	89	2939	7.272156237E+00	2.10E-05

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
	LFBGSE	92	94	2942	7.272155962E+00	1.27E-05
OBSTCLBM $n = 10000$	LRHB-qWolfe	81	85	2938	7.272155944E+00	2.59E-05
	LRHB-qArmijo	85	90	2939	7.272155971E+00	1.21E-05
	LFBGSE	83	84	2944	7.272155947E+00	4.71E-05
OBSTCLBU $n = 10000$	LRHB-qWolfe	89	92	2939	7.272155983E+00	1.37E-05
	LRHB-qArmijo	103	136	2940	7.272156279E+00	3.06E-05
	LFBGSE	85	87	2944	7.272156056E+00	1.09E-05
ODC $n = 5184$	LRHB-qWolfe	174	175	284	-1.137175556E-02	6.12E-06
	LRHB-qArmijo	211	215	284	-1.137178193E-02	7.17E-06
	LFBGSE	175	178	284	-1.137177649E-02	4.38E-06
ODNAMUR $n = 11130$	LRHB-qWolfe	11864	15140	4725	9.239435608E+03	8.27E-02
	LRHB-qArmijo	11521	13714	4730	9.239217617E+03	8.26E-02
	LFBGSE	11856	12225	5536	9.237928627E+03	5.60E-02
OSLBQP $n = 8$	LRHB-qWolfe	1	2	2	6.250000000E+00	0.00E+00
	LRHB-qArmijo	1	2	2	6.250000000E+00	0.00E+00
	LFBGSE	2	3	3	6.250000000E+00	0.00E+00
PALMER1 $n = 4$	LRHB-qWolfe	14	29	0	1.175460254E+04	8.36E-05
	LRHB-qArmijo	30	67	0	1.175460254E+04	4.43E-05
	LFBGSE	15	32	0	1.175460254E+04	5.51E-02
PALMER1A $n = 6$	LRHB-qWolfe	62	82	0	8.988305837E-02	2.72E-06
	LRHB-qArmijo	57	89	0	8.988305837E-02	3.90E-06
	LFBGSE	768	889	0	8.988305838E-02	7.42E-06
PALMER1B $n = 4$	LRHB-qWolfe	32	46	0	3.447349483E+00	5.79E-07
	LRHB-qArmijo	40	69	0	3.447349483E+00	2.67E-05
	LFBGSE	52	70	0	3.447349483E+00	3.01E-05
PALMER1E $n = 8$	LRHB-qWolfe	293	343	0	8.352321576E-04	5.22E-06
	LRHB-qArmijo	2228	2698	0	8.352321575E-04	2.96E-07
	LFBGSE	-- ⁱ	-- ⁱ	--	--	--
PALMER2 $n = 4$	LRHB-qWolfe	38	62	0	3.651097532E+03	3.99E-03
	LRHB-qArmijo	36	58	0	3.651097532E+03	1.43E-02
	LFBGSE	16	29	0	3.651097532E+03	2.98E-03
PALMER2A $n = 6$	LRHB-qWolfe	80	106	0	1.710971650E-02	6.10E-06
	LRHB-qArmijo	209	350	0	1.710971650E-02	1.62E-06
	LFBGSE	681	772	0	1.710971650E-02	5.14E-06
PALMER2B $n = 4$	LRHB-qWolfe	22	34	0	6.232669042E-01	3.17E-06
	LRHB-qArmijo	30	57	0	6.232669042E-01	4.57E-07
	LFBGSE	31	46	0	6.232669042E-01	1.17E-05
PALMER2E $n = 8$	LRHB-qWolfe	2124	2366	0	2.065109225E-04	9.35E-06
	LRHB-qArmijo	278	454	0	2.065035102E-04	7.69E-06
	LFBGSE	-- ⁱ	-- ⁱ	--	--	--
PALMER3 $n = 4$	LRHB-qWolfe	10	24	1	2.416983404E+03	8.35E-03
	LRHB-qArmijo	31	51	0	2.265958220E+03	2.16E-02
	LFBGSE	8	11	1	2.416980643E+03	9.82E-05
PALMER3A $n = 6$	LRHB-qWolfe	110	148	0	2.043142570E-02	1.62E-06
	LRHB-qArmijo	264	495	0	2.043142570E-02	6.82E-06
	LFBGSE	499	581	0	2.043142570E-02	1.68E-06

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
PALMER3B $n = 4$	LRHB-qWolfe	73	98	0	4.227647275E+00	2.93E-06
	LRHB-qArmijo	19	32	0	4.227647275E+00	1.70E-05
	LBFGBS	32	48	0	4.227647275E+00	1.36E-05
PALMER3E $n = 8$	LRHB-qWolfe	160	186	0	5.074105330E-05	4.69E-06
	LRHB-qArmijo	1399	1635	0	5.074203154E-05	2.31E-06
	LBFGBS	-- ⁱ	-- ⁱ	--	--	--
PALMER4 $n = 4$	LRHB-qWolfe	13	27	1	2.424016445E+03	5.85E-03
	LRHB-qArmijo	24	44	0	2.285383227E+03	2.88E-03
	LBFGBS	8	11	1	2.424016730E+03	8.69E-05
PALMER4A $n = 6$	LRHB-qWolfe	80	105	0	4.060614092E-02	5.03E-07
	LRHB-qArmijo	60	80	0	4.060614092E-02	6.63E-06
	LBFGBS	284	337	0	4.060614092E-02	3.59E-06
PALMER4B $n = 4$	LRHB-qWolfe	49	75	0	6.835138635E+00	1.35E-06
	LRHB-qArmijo	23	39	0	6.835138635E+00	7.23E-06
	LBFGBS	22	37	0	6.835138635E+00	5.83E-05
PALMER4E $n = 8$	LRHB-qWolfe	143	155	0	1.480034745E-04	9.47E-06
	LRHB-qArmijo	1396	1669	0	1.480034750E-04	3.32E-06
	LBFGBS	-- ⁱ	-- ⁱ	--	--	--
PALMER5A $n = 8$	LRHB-qWolfe	75	87	0	2.128086548E+00	2.85E-06
	LRHB-qArmijo	179	221	0	2.128086643E+00	4.33E-06
	LBFGBS	-- ^a	-- ^a	--	--	--
PALMER5B $n = 9$	LRHB-qWolfe	807	1076	0	9.752418226E-03	9.63E-07
	LRHB-qArmijo	525	739	0	1.512963479E-02	6.49E-06
	LBFGBS	-- ^a	-- ^a	--	--	--
PALMER5E $n = 8$	LRHB-qWolfe	-- ^a	-- ^a	--	--	--
	LRHB-qArmijo	31711	63388	0	2.071593905E-02	3.36E-07
	LBFGBS	3902	4943	0	2.521923875E-02	4.13E-06
PALMER6A $n = 6$	LRHB-qWolfe	149	191	0	5.594885356E-02	1.57E-06
	LRHB-qArmijo	344	626	0	5.594885357E-02	3.11E-06
	LBFGBS	643	781	0	5.594885362E-02	9.38E-06
PALMER6E $n = 8$	LRHB-qWolfe	132	152	0	2.239541033E-04	1.99E-06
	LRHB-qArmijo	2750	3301	0	2.239541198E-04	7.06E-06
	LBFGBS	-- ⁱ	-- ⁱ	--	--	--
PALMER7A $n = 6$	LRHB-qWolfe	14	17	0	2.792937845E+01	2.88E-05
	LRHB-qArmijo	14	17	0	2.792939388E+01	2.36E-04
	LBFGBS	-- ^a	-- ^a	--	--	--
PALMER7E $n = 8$	LRHB-qWolfe	7896	10658	0	1.015389960E+01	4.21E-05
	LRHB-qArmijo	5483	9940	0	1.015389921E+01	1.11E-04
	LBFGBS	6484	7543	0	1.015399779E+01	1.07E-04
PALMER8A $n = 6$	LRHB-qWolfe	110	165	0	7.400969864E-02	4.90E-06
	LRHB-qArmijo	19	24	0	6.969709610E+00	1.47E-05
	LBFGBS	337	404	0	7.400969864E-02	6.21E-06
PALMER8E $n = 8$	LRHB-qWolfe	150	171	0	6.339306147E-03	3.13E-06
	LRHB-qArmijo	1789	2126	0	6.339306147E-03	7.05E-06
	LBFGBS	-- ⁱ	-- ⁱ	--	--	--
PENTDI	LRHB-qWolfe	2	4	2502	-7.500000000E-01	0.00E+00

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
$n = 5000$	LRHB-qArmijo	2	4	2502	-7.500000000E-01	8.88E-16
	LFGS	1	3	4998	-7.500000000E-01	0.00E+00
PFIT1LS $n = 3$	LRHB-qWolfe	419	574	0	2.000546438E-14	3.94E-08
	LRHB-qArmijo	873	1702	0	6.831651528E-16	2.24E-06
LFGS	LRHB-qWolfe	199	269	0	1.090915502E-15	2.73E-07
	LRHB-qArmijo	1452	1997	0	1.728775312E-16	2.61E-07
PFIT2LS $n = 3$	LRHB-qWolfe	3202	6362	0	9.513499205E-16	3.46E-06
	LFGS	527	720	0	1.057556002E-18	4.46E-08
PFIT3LS $n = 3$	LRHB-qWolfe	531	726	0	4.184870652E-15	1.86E-06
	LRHB-qArmijo	4546	9048	0	8.392584129E-16	5.00E-06
LFGS	LRHB-qWolfe	523	692	0	1.031754019E-15	7.39E-06
	LRHB-qArmijo	1462	2025	0	3.449363183E-18	5.01E-07
PFIT4LS $n = 3$	LRHB-qWolfe	2561	5041	0	2.523672910E-17	2.60E-07
	LFGS	687	914	0	1.163234756E-14	4.88E-06
POWELLBC $n = 1000$	LRHB-qWolfe	1350	1606	205	3.108635002E+05	3.00E+00
	LRHB-qArmijo	1802	2077	221	3.118664669E+05	2.71E+00
LFGS	LRHB-qWolfe	1	3	1000	2.323322948E+06	1.00E+00
	LRHB-qWolfe	-- ^a	-- ^a	--	--	--
PROBPENL $n = 500$	LRHB-qArmijo	3	5	0	3.991983927E-07	1.99E-07
	LFGS	2	4	0	3.991983927E-07	1.99E-07
PSPDOC $n = 4$	LRHB-qWolfe	10	11	1	2.414213562E+00	5.99E-06
	LRHB-qArmijo	13	18	1	2.414213562E+00	4.08E-07
LFGS	LRHB-qWolfe	11	12	1	2.414213562E+00	4.99E-07
	LRHB-qWolfe	97661	103142	0	1.935757657E-06	9.92E-06
QR3DLS $n = 610$	LRHB-qArmijo	88513	103087	0	6.348844034E-07	9.50E-06
	LFGS	51241	56607	0	1.403379963E-05	9.95E-06
QRTQUAD $n = 5000$	LRHB-qWolfe	63	127	0	-2.648158493E+11	3.56E+05
	LRHB-qArmijo	32	75	0	-1.106548080E+11	9.74E+04
LFGS	LRHB-qWolfe	13	26	0	-2.648112175E+11	6.70E+02
	LRHB-qWolfe	3	7	4999	-1.250000000E+09	0.00E+00
QUDLIN $n = 5000$	LRHB-qArmijo	24	54	4975	-1.249990902E+09	2.31E+02
	LFGS	1	2	5000	-1.250000000E+09	0.00E+00
RAYBENDL $n = 2050$	LRHB-qWolfe	10049	12423	4	9.624244457E+01	6.46E-04
	LRHB-qArmijo	-- ^a	-- ^a	--	--	--
LFGS	LRHB-qWolfe	5670	7316	4	9.624284358E+01	4.03E-04
	LRHB-qWolfe	3639	3817	4	9.624173065E+01	4.04E-04
RAYBENDS $n = 2050$	LRHB-qArmijo	3618	4687	4	9.624212502E+01	3.24E-04
	LFGS	2777	2987	4	9.624184286E+01	1.30E-04
S368 $n = 8$	LRHB-qWolfe	10	11	2	-7.500000000E-01	6.29E-06
	LRHB-qArmijo	12	14	2	-7.500000000E-01	4.31E-06
LFGS	LRHB-qWolfe	11	12	2	-7.500000000E-01	2.27E-07
	LRHB-qWolfe	172	179	0	1.224442839E-05	8.95E-06
SANTALS $n = 21$	LRHB-qArmijo	188	212	0	1.224534661E-05	9.77E-06
	LFGS	235	247	0	1.230811893E-05	8.19E-06
SCOND1LS $n = 5002$	LRHB-qWolfe	390108	391126	2	5.110926060E-02	8.98E-06
	LRHB-qArmijo	433173	444810	2	5.963186799E-02	7.30E-06

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
	LFBGSE	365277	377391	2	6.048991166E-02	1.01E-05
SIM2BQP $n = 2$	LRHB-qWolfe	1	2	1	0.000000000E+00	0.00E+00
	LRHB-qArmijo	2	4	1	0.000000000E+00	0.00E+00
	LFBGSE	1	2	2	0.000000000E+00	0.00E+00
SIMBQP $n = 2$	LRHB-qWolfe	3	5	1	6.162975822E-32	1.11E-15
	LRHB-qArmijo	6	9	1	1.354169492E-34	5.20E-17
	LFBGSE	4	6	1	9.183549616E-42	1.36E-20
SINEALI $n = 1000$	LRHB-qWolfe	25	33	0	-9.990096154E+04	7.50E-02
	LRHB-qArmijo	35	49	0	-9.990096157E+04	6.13E-02
	LFBGSE	12	21	0	-9.987336277E+04	3.50E-02
SPECAN $n = 9$	LRHB-qWolfe	47	51	0	1.647283700E-13	5.57E-07
	LRHB-qArmijo	74	95	0	1.646622264E-13	7.86E-07
	LFBGSE	137	152	0	2.143021540E-13	9.28E-06
SSC $n = 5184$	LRHB-qWolfe	127	129	284	-2.078173251E+00	8.94E-06
	LRHB-qArmijo	172	177	284	-2.078173174E+00	2.08E-05
	LFBGSE	124	128	284	-2.078173237E+00	5.05E-06
TORSION1 $n = 5476$	LRHB-qWolfe	113	115	1912	-4.302757850E-01	4.04E-06
	LRHB-qArmijo	82	85	1912	-4.302757676E-01	8.08E-06
	LFBGSE	104	108	1916	-4.302757911E-01	3.95E-06
TORSION2 $n = 5476$	LRHB-qWolfe	137	139	1912	-4.302757695E-01	7.59E-06
	LRHB-qArmijo	110	112	1912	-4.302757876E-01	4.00E-06
	LFBGSE	116	119	1916	-4.302757560E-01	3.73E-06
TORSION3 $n = 5476$	LRHB-qWolfe	47	49	3672	-1.216956071E+00	9.57E-06
	LRHB-qArmijo	46	48	3672	-1.216956075E+00	9.94E-06
	LFBGSE	51	53	3676	-1.216956054E+00	9.93E-06
TORSION4 $n = 5476$	LRHB-qWolfe	70	73	3672	-1.216956078E+00	4.52E-06
	LRHB-qArmijo	62	71	3672	-1.216956052E+00	1.93E-05
	LFBGSE	77	81	3676	-1.216956075E+00	5.91E-06
TORSION5 $n = 5476$	LRHB-qWolfe	23	26	4520	-2.863377959E+00	1.46E-05
	LRHB-qArmijo	24	26	4520	-2.863377962E+00	2.48E-05
	LFBGSE	24	26	4524	-2.863377962E+00	1.37E-05
TORSION6 $n = 5476$	LRHB-qWolfe	36	41	4520	-2.863377964E+00	1.02E-05
	LRHB-qArmijo	41	50	4520	-2.863377961E+00	1.29E-05
	LFBGSE	44	46	4524	-2.863377963E+00	1.49E-05
TORSIONA $n = 5476$	LRHB-qWolfe	110	112	1848	-4.182961369E-01	3.50E-06
	LRHB-qArmijo	103	110	1848	-4.182961276E-01	7.23E-06
	LFBGSE	126	131	1852	-4.182961459E-01	4.00E-06
TORSIONB $n = 5476$	LRHB-qWolfe	131	133	1848	-4.182961442E-01	8.50E-06
	LRHB-qArmijo	130	131	1848	-4.182961230E-01	3.68E-06
	LFBGSE	129	133	1852	-4.182961439E-01	3.90E-06
TORSIONC $n = 5476$	LRHB-qWolfe	49	51	3640	-1.204208936E+00	5.29E-06
	LRHB-qArmijo	56	60	3640	-1.204208939E+00	6.73E-06
	LFBGSE	51	53	3644	-1.204208921E+00	8.31E-06
TORSIOND $n = 5476$	LRHB-qWolfe	58	61	3640	-1.204208940E+00	5.41E-06
	LRHB-qArmijo	66	70	3640	-1.204208938E+00	9.73E-06
	LFBGSE	77	79	3644	-1.204208937E+00	1.59E-05

Table 2: Final results and statistics from three solvers. (Continued)

Problem	Solver	nItn	Nf	$\mathcal{A}(x_T)$	Obj	Grad
TORSIONE $n = 5476$	LRHB-qWolfe	24	26	4504	-2.850247856E+00	1.16E-05
	LRHB-qArmijo	24	27	4504	-2.850247852E+00	1.59E-05
	LBFGB	24	26	4508	-2.850247856E+00	9.92E-06
TORSIONF $n = 5476$	LRHB-qWolfe	37	41	4504	-2.850247850E+00	1.57E-05
	LRHB-qArmijo	39	49	4504	-2.850247853E+00	1.71E-05
	LBFGB	42	43	4508	-2.850247855E+00	1.12E-05
WALL10 $n = 1461$	LRHB-qWolfe	-- ^a	-- ^a	--	--	--
	LRHB-qArmijo	-- ^a	-- ^a	--	--	--
	LBFGB	-- ^a	-- ^a	--	--	--
WALL100 $n = 149624$	LRHB-qWolfe	4249	4265	38904	-8.954382962E+03	3.09E-02
	LRHB-qArmijo	4212	4345	38907	-8.954350885E+03	5.86E-02
	LBFGB	4478	4607	38905	-8.954364491E+03	3.16E-02
WALL20 $n = 5924$	LRHB-qWolfe	34001	34116	537	-9.504603323E+05	8.00E+00
	LRHB-qArmijo	67916	70803	787	-7.108290084E+05	3.79E+00
	LBFGB	242125	249125	1193	-3.298069679E+05	1.06E+00
WALL50 $n = 37311$	LRHB-qWolfe	-- ^t	-- ^t	--	--	--
	LRHB-qArmijo	-- ^t	-- ^t	--	--	--
	LBFGB	-- ^t	-- ^t	--	--	--
WEEDS $n = 3$	LRHB-qWolfe	41	62	0	2.587277395E+00	1.58E-05
	LRHB-qArmijo	-- ^a	-- ^a	--	--	--
	LBFGB	37	66	0	2.587277395E+00	2.51E-08
YFIT $n = 3$	LRHB-qWolfe	60	84	0	6.669822755E-13	1.46E-06
	LRHB-qArmijo	142	260	0	6.669739805E-13	3.95E-07
	LBFGB	75	91	0	6.690888560E-13	1.51E-06

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