4.1. (20 points) Consider the following initial value problem:

\[
\begin{aligned}
  x'(t) &= f(t, x(t)) \\
  x(0) &= 1.
\end{aligned}
\]

(a) Let \( t_i = ih \), where \( h > 0 \) is the stepsize. Integrate \( x'(t) = f(t, x(t)) \) over \([t_{n-3}, t_n]\) to write \( x_n - x_{n-3} \) in terms of an integral of \( f(t, x(t)) \).

(b) Let \( f_i = f(t_i, x(t_i)) \). Find the interpolation formula in Lagrange form using the three points \((t_{n-3}, f_{n-3}), (t_{n-2}, f_{n-2})\) and \((t_{n-1}, f_{n-1})\).

(c) Approximate the integral of \( f(t, x(t)) \) you got from part (a) using the interpolation polynomial you got from part (b). Explain very briefly why such approximation is exact if \( f(t, x(t)) \) is a polynomial of degree 2 in \( t \).

(d) With the result of (c), suggest a new multi-step method. Analyze its consistency.

(e) Consider the following initial-value problem:

\[
\begin{aligned}
  x'(t) &= x(t) \quad 0 \leq t \leq 1 \\
  x(0) &= 1.
\end{aligned}
\]

Try to approximate the solution with the new multi-step method with stepsize 0.1 with initial values \( x(0) = 1.000, x(0.1) = 1.105, x(0.2) = 1.221 \). Tabulate your result.

4.2. (15 points) Consider the following multi-step method:

\[
y_{n+3} - y_{n+2} = \frac{1}{12} h[23f_{n+2} - af_{n+1} + 5f_n],
\]

where \( a \in \mathbb{R} \) is a constant and \( h > 0 \) is the stepsize.

(a) There is a very simple reason for the inconsistency of the scheme if \( a \neq 16 \). Please explain briefly.

(b) Evaluate the order of the above scheme if \( a = 16 \) by directly expanding the local truncation error \( \tau(t, y, h) \) up to a suitable order of \( h \).

(c) We can see that the above scheme is consistent if and only if it converges (which is not the case for a general multi-step method). Please, explain with the aid of theorems presented in the lecture notes.

4.3. (15 points) Consider the following multi-step method:

\[
a^2x_{n+2} - ax_{n+1} - 12x_n = 7ahf_{n+1},
\]

where \( a \in \mathbb{R} \) is a constant and \( h > 0 \) is the stepsize.
(a) Evaluate directly by definition the local truncation error \( \tau(t, x, h) \) in the form \( d_0 x(t_n) + d_1 h x'(t_n) + d_2 h^2 x''(t_n) + O(h^3) \), where \( d_i \) depend on \( a \).

(b) If the above scheme is consistent, find the possible values of \( a \) directly from the local truncation error.

(c) Evaluate the order of the above scheme for the suitable values of \( a \) such that the scheme is consistent. You may need to expand more terms for the local truncation error to get the result if necessary.

(d) Discuss the stability of the scheme for such \( a \).

4.4. (10 points) From the textbook: §8.6, Questions 1,4

(a) Find the general solution of the system

\[
\begin{align*}
x' &= 3x - 4y + e^t \\
y' &= x - y - e^t
\end{align*}
\]

Hint: Try functions of the form \( e^t \), \( te^t \), and \( t^2 e^t \).

(b) Convert the system of second-order ordinary differential equations

\[
\begin{align*}
x'' - x' y &= 3y' x \log t \\
y'' - 2xy' &= 5x' y \sin t
\end{align*}
\]

into a system of first-order equations in which \( t \) does not appear explicitly.

4.5. (15 points) (MATLAB §8.6, Question 4)

Write a subprogram or procedure that takes one step of length \( h \) in the fourth-order Runge-Kutta procedure. Make it capable of handling a system of \( n \) differential equations with \( n \leq 20 \). Test your program by solving the following system on the interval \( 1 \leq t \leq 2 \). Use \( h = -0.01 \).

\[
\begin{align*}
x' &= x^{-2} + \log y + t^2 \\
y' &= e^y - \cos x + (\sin t)x - (xy)^{-3} \\
x(2) &= -2 \\
y(2) &= 1
\end{align*}
\]