We have
\[ \hat{M} \cdot \hat{M} = (\hat{M} - i \hat{\pi}_z) (\hat{M} + i \hat{\pi}_z) = \hat{M}^2 + \hat{\pi}_z^2 + i [\hat{M}_z, \hat{M}_z] = \hat{M}^2 - \hat{\pi}_z^2 - \hbar \hat{\pi}_z. \]  
(7.10.37)

If the operator \( \hat{M} \cdot \hat{M} \) acts on \( |\lambda, m_{\text{max}}\rangle \), it follows by using (7.10.31) that
\[ (\hat{M}^2 - \hat{\pi}_z^2 - h \hat{\pi}_z) |\lambda, m_{\text{max}}\rangle = \hat{M} \cdot \hat{M} \cdot |\lambda, m_{\text{max}}\rangle = 0. \]  
(7.10.38)

Therefore, by (7.10.26) and (7.10.27),
\[ (\lambda - m_{\text{max}}^2 - m_{\text{min}}^2) |\lambda, m_{\text{max}}\rangle = 0 \]
or
\[ \lambda = m_{\text{max}}^2 + m_{\text{max}}. \]  
(7.10.39)

Similarly,
\[ \hat{M} \cdot \hat{M} = \hat{M}^2 - \hat{\pi}_z^2 + h \hat{\pi}_z. \]  
(7.10.40)

If this operator acts on \( |\lambda, m_{\text{max}}\rangle \), and (7.10.34) is used, we obtain
\[ \lambda - m_{\text{min}}^2 + m_{\text{min}} = 0. \]  
(7.10.41)

If we equate the two results for \( \lambda \) from (7.10.39) and (7.10.41), it turns out that
\[ (m_{\text{max}} + m_{\text{min}})(m_{\text{min}} - m_{\text{max}} - 1) = 0. \]  
(7.10.42)

Thus
\[ m_{\text{max}} = -m_{\text{min}}. \]  
(7.10.43)

Therefore the admissible values of \( m \) lie symmetrically about the origin. Since the extreme values differ by an integer, it follows that
\[ m_{\text{max}} - m_{\text{min}} = 2l. \]  
(7.10.44)

where
\[ l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots. \]  
(7.10.45)

These results combined with (7.10.43) show that
\[ -l \leq m \leq l \]  
(2l + 1 values).

Finally, it follows from (7.10.39) and (7.10.44) that
\[ \lambda = (l+1), \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots. \]  
(7.10.47)

This is a definite proof for integer and half-integer eigenvalues for the angular momentum. Particles with integral spin are called the Bosons, those with half-integral spins are known as Fermions.

The two different kinds of angular momentum operators can be combined to define the total angular momentum
\[ \hat{J} = \hat{L} + \hat{\pi}_z \]  
(7.10.48)

with the components \( \hat{J}_x = \hat{L}_x + \hat{\pi}_z, \hat{J}_y = \hat{L}_y + \hat{\pi}_z, \hat{J}_z = \hat{L}_z + \hat{\pi}_z \).

It follows from the properties of \( \hat{L} \) and \( \hat{\pi}_z \) that \( \hat{J} \) satisfies the usual commutation relations
\[ [\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z, \]  
(7.10.49a)
\[ [\hat{J}_x, \hat{J}_z] = i\hbar \hat{J}_y, \]  
(7.10.49b)
\[ [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x, \]  
(7.10.49c)

and hence
\[ [\hat{J}_x, \hat{J}^2] = [\hat{J}_y, \hat{J}^2] = [\hat{J}_z, \hat{J}^2] = 0, \]  
(7.10.50abc)

where
\[ \hat{J}^2 = \hat{J}^2_x + \hat{J}^2_y + \hat{J}^2_z. \]  
(7.10.51)

It can readily be shown that
\[ \hat{J}^2|l, m\rangle = (l(l+1) \hbar^2)|l, m\rangle, \]  
(7.10.52)
\[ \hat{J}_z|l, m\rangle = \hbar m|l, m\rangle. \]  
(7.10.53)

This means that the eigenvalues of \( \hat{J}^2 \) and \( \hat{J}_z \) are \( l(l+1) \hbar^2 \) and \( \hbar m \), respectively, where \( |m| \leq l \) and the quantum numbers may be either integers or half-integers.

Finally, it follows that
\[ [\hat{J}_l, \hat{L}_z] = 2\hat{M}_z[\hat{L}_z, \hat{L}_z] + 2\hat{M}_l[\hat{L}_z, \hat{L}_z] = \hbar[\hat{L}_z, \hat{L}_z], \]  
(7.10.54)
\[ [\hat{J}_l, \hat{M}_z] = -\hbar[\hat{L}_z, \hat{L}_z]. \]  
(7.10.55)

7.11. Exercises

(a) Use the Lagrangian, \( L = \frac{1}{2}m(r^2 + \dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(z^2 + y^2 + x^2) \) for the three-dimensional isotropic harmonic oscillator, and Lagrange’s equations of motion to show that the total energy is constant where \( k \) is the force constant.

(b) Show that the Lagrangian for the oscillator in spherical polar coordinates \( (r, \theta, \phi) \) is
\[ L = T - V = \frac{1}{2}m(r^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - \frac{1}{2}kr^2, \]
where \( k = \frac{4}{3}m\omega^2 \).

Hence write down the Lagrange equations of motion.
(2) Consider a single particle of mass $m$ moving in a plane under a conservative force with potential $V(r)$, where $r$ is distance from the origin of coordinates. With $r$ and $\theta$ as generalized coordinates describing the motion of the particle, show that the corresponding momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \ddot{\theta},$$

where $L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$. Hence show that

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r), \quad mr^2 \dot{\theta} = \text{constant}, \quad m(r - r\dot{\theta}) = \frac{\partial V}{\partial r}.$$

Give an interpretation of each of the above results.

(3) If $A$ is a complex dynamical function of $q$ and $p$, $A^*$ is its complex conjugate, and if the Poisson bracket \{A, A^*\} = i, compute \{A, A^*\}, \{A^*, A^*\}, \{A^*, A A^*\}, and \{A^*, AA^*\}.

(4) Find the Hamiltonian and Hamilton’s equations of motion for

(i) The simple harmonic oscillator, $T = \frac{1}{2} m \dot{x}^2$ and $V = \frac{1}{2} k x^2$

(ii) The planetary motions, $T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$, and $V = mu / (2a - 1/r)$.

In this case, derive the differential equations for the central orbit.

(5) Establish the following results for the Poisson brackets:

(i) \{A, B\} = \{B, A\},

(ii) \{(A + B), C\} = \{A, C\} + \{B, C\},

(iii) \{AB, C\} = \{A, C\} B + A\{B, C\},

(iv) \{A, \alpha\} = 0,

(v) \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 (Jacobi’s Identity),

where $A, B, C$ are canonical functions and $\alpha$ is a scalar.

(6) Show that

(i) \{\hat{A}, \hat{B}\} = -\{\hat{B}, \hat{A}\},

(ii) \{\hat{A} + \hat{B}, \hat{C}\} = \{\hat{A}, \hat{C}\} + \{\hat{B}, \hat{C}\},

(iii) \{\hat{A}, \hat{B} + \hat{C}\} = \{\hat{A}, \hat{B}\} + \{\hat{A}, \hat{C}\},

(iv) \{\hat{A}\hat{B}, \hat{C}\} = \{\hat{A}, \hat{C}\} \hat{B} + \hat{A}\{\hat{B}, \hat{C}\},

(v) \{\hat{A}, \hat{B}\hat{C}\} = \{\hat{A}, \hat{B}\} \hat{C} + \hat{B}\{\hat{A}, \hat{C}\},

(vi) \{\hat{A}, \{\hat{B}, \hat{C}\}\} + \{\hat{B}, \{\hat{C}, \hat{A}\}\} + \{\hat{C}, \{\hat{A}, \hat{B}\}\} = 0 (Jacobi’s Identity),

(vii) \{\hat{A}^2, \hat{B}\} = A\{\hat{A}, \hat{B}\} + \{\hat{A}, \hat{B}\} A,$

(viii) \{\hat{A}, \alpha\} = 0, \alpha \text{ is a scalar.}

(7) For the three dimensional position and momentum operators of a particle, prove that

$$[\hat{r}_i, \hat{p}_j] = i\hbar \delta_{ij},$$

where the suffixes $i, j$ take the values 1, 2, 3 for the $x, y, z$ components of $\hat{r}$ and $\hat{p}$, respectively.

(8) By direct evaluation for canonically conjugate variables $q$ and $p$, show that

(i) $[p^i, p^j] = 2\hbar \delta^{ij}$

(ii) $[p^i, q^j] = -2i\hbar \delta^{ij}$

(iii) $[\hat{p}_x, \hat{p}_y] = 2\hbar \delta xy$

(iv) $[\hat{p}_x, \hat{p}_z] = -2i\hbar \delta xz$

(9) If $A$ and $B$ are any two operators which both commute with their commutator $[\hat{A}, \hat{B}]$ prove that

$$[\hat{A}, \hat{B}^*] = n\hat{B}^{*\dagger} [\hat{A}, \hat{B}],$$

$$[\hat{A}^*, \hat{B}] = n\hat{A}^{*\dagger} [\hat{A}, \hat{B}].$$

(10) Establish the following commutator relations:

$[\hat{L}_x, \hat{L}_z] = [\hat{L}_z, \hat{L}_y] = [\hat{L}_y, \hat{L}_x] = 0.$

(11) Show that

$[\hat{\mathcal{L}}_x, \hat{\mathcal{L}}_z] = \hbar \hat{\mathcal{L}}_x,$

$[\hat{\mathcal{L}}_y, \hat{\mathcal{L}}_z] = -\hbar \hat{\mathcal{L}}_y,$

$[\hat{\mathcal{L}}_z, \hat{\mathcal{L}}_x] = \hbar \hat{\mathcal{L}}_z.$

$[\hat{L}_z, \hat{L}_x] = \hbar \hat{L}_z,$

$[\hat{L}_y, \hat{L}_x] = \hbar \hat{L}_y.$

$[\hat{L}_x, \hat{L}_y] = 0.$

(12) Prove that

$$\hat{J}^2 = \hat{L}_x^2 + \hat{L}_y^2 + 2\hat{L}_z \cdot \hat{\mathcal{M}} = \hat{L}_x^2 + \hat{L}_y^2 + 2\hat{L}_z \cdot \hat{\mathcal{M}},$$

$$2\hat{L} \cdot \hat{\mathcal{M}} = \hat{J}^2 - \hat{L}_z^2 - \hat{\mathcal{M}}^2.$$
(13) Show that the probability for a position measurement on the state $\Psi(x, t)$ to yield a value somewhere between $x_1$ and $x_2$ is

$$P(x_1, x_2, t) = \int_{x_1}^{x_2} |\Psi(x, t)|^2 \, dx.$$ 

Using the Schrödinger equations, derive the result

$$\frac{d}{dt} P(x_1, x_2, t) = J(x_1, t) - J(x_2, t),$$

where

$$J(x, t) = \frac{i\hbar}{2m} \left[ \Psi \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial \Psi}{\partial x} \right].$$

(14) Use the inner product

$$\langle \phi, \psi \rangle = \int_{-\infty}^{\infty} \bar{\phi} \psi \, dx,$$

and the property $\langle \phi, \psi \rangle \to (0, 0)$ as $|x| \to \infty$, to show that the position operator $\hat{x} = x$, the momentum operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$, and the energy operator $\hat{H} = \hat{p}^2/2m + V(x)$ are Hermitian operators.

(15) Establish the following commutation relations for the orbital angular momentum operators:

$$[\hat{L}_x, \hat{\theta}] = 0,$$
$$[\hat{L}_y, \hat{\phi}] = i\hbar \hat{\theta},$$
$$[\hat{L}_z, \hat{\phi}] = -i\hbar \hat{\phi},$$
$$[\hat{L}_x, \hat{\hat{p}}] = 0,$$
$$[\hat{L}_y, \hat{\hat{p}}] = i\hbar \hat{p}_y,$$
$$[\hat{L}_z, \hat{\hat{p}}] = -i\hbar \hat{p}_z.$$ 

(16) Prove the Heisenberg uncertainty relation for the harmonic oscillator

$$\Delta x \Delta p \geq \frac{1}{2} \hbar.$$ 

(17) If $\hat{A}$ and $\hat{B}$ are constants of motion, show that the commutator $[\hat{A}, \hat{B}]$ is also a constant of motion.

(18) Show that, for the linear harmonic oscillator,

$$[\hat{H}, \hat{A}] = (\hbar \omega) \hat{A},$$
$$[\hat{H}, \hat{A}^*] = (\hbar \omega) \hat{A}^*,$$

where

$$\hat{A} = \hat{x} / \sqrt{\hbar \omega} \quad \text{and} \quad \hat{A}^* = \hat{a}^* / \sqrt{\hbar \omega}.$$ 

(19) For the three dimensional anisotropic Planck's oscillator, the Hamiltonian is given by

$$H = \frac{1}{2m} \hat{p}_1^2 + \frac{1}{2} m \omega_1^2 \hat{x}_1^2,$$

so that total Hamiltonian $H = H_1 + H_2 + H_3$ and the total energy $E = E_1 + E_2 + E_3$, where $E_1$, $E_2$, $E_3$ are energies of each of the independent degrees of freedom. Show that

$$E = (n_1 + \frac{3}{2}) \hbar \omega_1 + (n_2 + \frac{3}{2}) \hbar \omega_2 + (n_3 + \frac{3}{2}) \hbar \omega_3.$$ 

In the case of an isotropic oscillator, $\omega_1 = \omega_2 = \omega_3 = \omega$, derive the result

$$E_N = (N + \frac{3}{2}) \hbar \omega, \quad N = n_1 + n_2 + n_3 = 0, 1, 2, 3, \ldots$$ 

(20) Prove the compatibility theorem which states that any one of the following conditions implies the other two:

(i) $\hat{A}$ and $\hat{B}$ are compatible,
(ii) $\hat{A}$ and $\hat{B}$ possess a common eigenbasis,
(iii) $\hat{A}$ and $\hat{B}$ commute,

where $A$ and $B$ are two observables with corresponding operators $\hat{A}$ and $\hat{B}$.

(21) If the eigenvectors $\{\phi_n(x)\}$ form an orthonormal basis in a Hilbert space, show that any state vector $\psi(x)$ satisfies the result

$$\langle \phi, \psi \rangle = \sum_{n=1}^{\infty} \langle \phi_n, \psi \rangle.$$ 

(22) If $\hat{A}' = \hat{A} - \langle \hat{A} \rangle$ and $\hat{B}' = \hat{B} - \langle \hat{B} \rangle$, prove the following results:

(i) $\hat{A}'$ and $\hat{B}'$ are Hermitian operators,
(ii) $[\hat{A}', \hat{B}'] = [\hat{A}, \hat{B}]$,
(iii) $\langle \hat{A}' \hat{A}' \hat{A}' \hat{A} \psi \rangle = (\Delta \hat{A})^2.$

Use these results to establish the generalized uncertainty relation.

(23) Using $\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A} \psi \, dx$ prove that the expectation values of position and momentum in the state $\Psi(x, t)$ are

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} x |\Psi(x, t)|^2 \, dx,$$
$$\langle \hat{p} \rangle = -i\hbar \int_{-\infty}^{\infty} \frac{\partial}{\partial x} |\Psi(x, t)|^2 \, dx.$$
Also show that
\[ \langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi(x, t)|^2 \, dx, \quad \langle \hat{p}^2 \rangle = \hbar^2 \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} \Psi(x, t) \right|^2 \, dx. \]

(24) Apply the basic commutation relations [\hat{x}, \hat{p}] = i\hbar \delta_y and rules of commutator algebra to show that
\[ [\hat{x}\hat{p}, \hat{H}] = \frac{i\hbar}{m} \hat{\beta}^2 + \hat{x}[\hat{\beta}, \hat{V}], \quad [\hat{p}\hat{x}, \hat{H}] = \frac{i\hbar}{m} \hat{\beta}^2 + \hat{p}[\hat{\beta}, \hat{V}], \]
\[ [\hat{p}\hat{p}, \hat{H}] = \frac{i\hbar}{m} \hat{\beta}^2 + \hat{p}[\hat{\beta}, \hat{V}]. \]

Hence combine them to obtain the Heisenberg equation of motion for the operator \( r \cdot \mathbf{p} \)
\[ \frac{d}{dt}(r \cdot \mathbf{p}) = \left( \frac{\mathbf{p} \cdot \nabla}{m} - (r \cdot \nabla V) \right). \]

Hence or otherwise prove the Virial Theorem for the stationary states:
\[ 2\langle T \rangle = (r \cdot \nabla V). \]

(25) Use the results in Exercise (9) for \( \hat{A} = \hat{x} \) and \( \hat{B} = \hat{p} \), to prove that for any Hamiltonian of the form
\[ \hat{H} = \frac{\hat{p}^2}{2m} + \alpha \hat{x}^2, \]
the following relation holds:
\[ [\hat{p}\hat{x}, \hat{H}] = i\hbar \left( \frac{\hat{p}^2}{m} - \alpha \hat{x}^2 \right) = i\hbar (2\hat{T} - \hat{p}\hat{V}). \]

(26) Use the Hamiltonian operator for the one dimensional simple harmonic oscillator in the form
\[ \hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{x}^2, \]
and then introduce the non-dimensional variables
\[ \hat{\chi} = \left( \frac{m\omega}{2\hbar} \right)^{1/2} \hat{x}, \quad \hat{\mathbf{p}} = \frac{1}{(2m\hbar\omega)^{1/2}} \hat{p}. \]

(a) Show that
(i) \( \hat{X} \) and \( \hat{P} \) are Hermitian operators,
(ii) \( \hat{H} = \hbar \omega (\hat{P}^2 + \hat{X}^2) \),
(iii) \( [\hat{X}, \hat{P}] = i \hbar \).